

# A. S. Kechris: Global Aspects of Ergodic Group Actions; Corrections and Updates (May 12, 2020)

**Page x, line 3:** Replace “the equivalence” by “an ergodic equivalence”.

**Page 8, last 7 lines of the proof of 2.5:** Replace  $\xi_i$  by  $\eta_i$ .

**Page 12, line 3-:** Add “.” at the end.

**Page 19, Theorem 3.13:** This was originally proved in H.A. Dye, On groups of measure preserving transformations, I, *Amer. J. Math.*, **81(1)** (1959), 119–159.

**Page 20, Theorem 4.1:** This was originally proved in H.A. Dye, On groups of measure preserving transformations, II, *Amer. J. Math.*, **85(4)** (1963), 551–576.

**Page 22, line 20-:** François Le Maître pointed out that in [BG1] the authors prove this result for the class of measure-class preserving equivalence relations but the older result for the case of measure preserving equivalence relations is due to H.A. Dye, On groups of measure preserving transformations, II, *Amer. J. Math.*, **85** (1963), 551–576, Proposition 5.1.

**Page 26:** An update concerning the results in the paragraph following 4.12:

Matui has shown that for ergodic hyperfinite  $E$ ,  $t([E]) = 2$ , and  $t([E_n]) \leq 2(n + 1)$ . Independently Marks also proved that  $t([E]) = 2$  and moreover showed that  $t([E_n]) \leq 2n$  and if  $E_n$  is induced by a free modular action of  $F_n$ , then  $t([E_n]) = n + 1$ .

Finally Le Maître proved that for *every ergodic* measure preserving  $E$  one has the formula  $t([E]) = \lfloor C_\mu(E) \rfloor + 1$ , where  $\lfloor C_\mu(E) \rfloor$  is the integer part of the cost of  $E$ , and therefore  $t([E_n]) = n + 1$ . He also obtained a related formula when  $E$  is aperiodic but not necessarily ergodic.

**Page 30, line 1-:** After “3.14” add “and 2.6”.

**Page 47, line 7:** “ $\ker(\pi_1) = \ker(\pi_2)$ ” should be “ $\ker(\bar{\pi}_1) = \ker(\bar{\pi}_2)$ ”.

**Page 67, line 18:** Delete “ergodic”.

**Page 71, 10.9:** Add in the statement of the theorem that  $\Gamma$  is an infinite countable group.

**Page 71, line 4:** Before “Let” add “By 10.8, we can assume that  $a$  is free”.

**Page 71, line 11:** Replace “The extreme ... Krein-Milman” by “Using the ergodic decomposition of  $a$  and the fact that  $a$  is free, we see that”.

**Page 71, line 13:** After “ergodic,” add “and non-atomic,”.

**Page 73, line 7-:** Replace “iff” by “ $\implies$ ”.

**Page 73, line 1-:** Replace “ $\Gamma \leq Z(\Delta)$ ” by “ $\Gamma \leq C_\Delta(T) =$  the centralizer of  $T$  in  $\Delta$ ”.

**Page 75, Remark:** Robin Tucker-Drob answered the question at the end of this remark by showing that  $C^*(\Gamma) = C^{**}(\Gamma)$  for all infinite groups  $\Gamma$ . The proof is as follows:

It is enough to show that if  $\Gamma$  has property (T), then  $C(\Gamma) = C^*(\Gamma) = C^{**}(\Gamma)$ . Indeed, suppose that  $\Gamma$  has property (T). Then  $\Gamma$  is finitely generated (see [BdlHV, 1.3.1]), so by Corollary 10.14 we have  $C(\Gamma) = C^*(\Gamma)$ . It remains to show that  $C(\Gamma) = C^{**}(\Gamma)$ . By Theorem 13.1 the set  $A$ , consisting of all actions  $a \in \text{FRERG}(\Gamma, X, \mu)$  with weakly dense conjugacy class in  $\text{FRERG}(\Gamma, X, \mu)$ , is dense  $G_\delta$  in  $(\text{FRERG}(\Gamma, X, \mu), w)$ . Fix  $a \in A$ . Let  $b \in \text{FR}(\Gamma, X, \mu)$  and let  $\mu = \int_Z \mu_z d\eta(z)$  be the ergodic decomposition of  $b$ , so  $\eta$ -almost every measure  $\mu_z$  is  $b$ -invariant and ergodic. Let  $b_z$  denote the action of  $\Gamma$  on  $(X, \mu_z)$ , so that  $b_z$  is almost surely ergodic. Since  $b$  is free,  $\eta$ -almost every  $b_z$  is free. Hence, for  $\eta$ -almost every  $z \in Z$  we have  $b_z \prec a$ . This implies  $b \prec i_Z \times a$ , where  $i_Z$  is the identity action of  $\Gamma$  on  $(Z, \eta)$ . Since  $\Gamma$  is finitely generated Corollary 10.14 gives  $C(i_Z \times a) \leq C(b)$ ; since  $C(i_Z \times a) = C(a)$ , we have  $C(a) \leq C(b)$ . As  $b \in \text{FR}(\Gamma, X, \mu)$  was arbitrary, this shows that  $C(a) = C(\Gamma)$ . Since  $a \in A$  was arbitrary and  $A$  is dense  $G_\delta$  in  $(\text{FRERG}(\Gamma, X, \mu), w)$ , this shows that  $C^{**}(\Gamma) = C(\Gamma)$ .

**Page 79, lines 4-5:** R. Tucker-Drob, Shift-minimal groups, fixed price 1, and the unique trace property, *arXiv: 1211.6395v3*, Corollary 6.22, shows that for infinite  $\Gamma$  and  $a \prec b$ ,  $a$  free:

$$C(b) < \infty \implies C(b) \leq C(a),$$

$$E_b \text{ treeable} \implies C(b) \leq C(a).$$

**Page 87, line 18:** Add “)” at the end of the formula.

**Page 90, 12.11:** Replace [HJ4] by [HJ5].

**Page 90, Proof of 13.1:** Here view (by picking representatives) the  $a_n, a$  as Borel actions and not equivalence classes of Borel actions up to equality a.e.

**Page 91, last Remark:** Add:

It is shown in D. Kerr and H. Li, *Ergodic Theory*, Springer, 2016, Theorem 4.45, using the results of D.S. Ornstein and B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, *J. Analyse Math.* **48** (1987), 1–141, that if  $\Gamma$  is amenable, then every action in  $\text{FR}(\Gamma, X, \mu)$  has dense conjugacy class in  $(\text{FR}(\Gamma, X, \mu), u)$ .

**Page 114, line 9-:** Replace “lemma” by “theorem”.

**Page 123, line 7-:** add “sofic” before  $\Gamma$ .

**Page 123, (IIIb) and (IVb):** G. Hjorth and A. Törnquist, The conjugacy relation on unitary representations, *Math. Res. Lett.*, **19(3)** (2012), 525–535, have shown that unitary equivalence for any  $\Gamma$  is Borel, in fact  $\mathbf{\Pi}_3^0$ .

In E. Gardella and M. Lupini, The complexity of conjugacy, orbit equivalence, and von Neumann equivalence of actions of nonamenable groups, *arXiv:1708.01327v2*, it is shown that isomorphism and orbit equivalence on free, ergodic (in fact even weak mixing) actions are not Borel for all non-amenable groups.

**Page 131, 3 lines before (B):** generically  $\rightarrow$  generally.

**Page 141, line 4 of the proof of 21.4:** add period.

**Page 184, last lines of 30.4, 30.5:**  $G$ -superrigid  $\rightarrow$   $G$ -cocycle superrigid.

**Page 185, third line of 30.6:**  $G$ -superrigid  $\rightarrow$   $G$ -cocycle superrigid.

**Page 186, 10-:**  $F_2$ -superrigid  $\rightarrow$   $F_2$ -cocycle superrigid.

**Page 186, 2-:**  $G$ -superrigid  $\rightarrow$   $G$ -cocycle superrigid.

**Page 208, line 7 of (B):** positive definite  $\rightarrow$  positive-definite.

**Page 214, line 14:** After “claim” add “applied to  $e_1, \dots, e_p$  and a given open set  $W' \subseteq W$ ”.

**Page 234:** add to index  
 $i_\Gamma$ , 67