Invariant uniformization and reducibility

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Abstract

Standard results in descriptive set theory provide sufficient conditions for a set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ to admit a Borel uniformization, namely, when P has "small" sections or "large" sections. We consider an invariant analogue of these results: Given a Borel equivalence relation E and an E-invariant set P with "small" or "large" sections, does P admit an E-invariant Borel uniformization?

Given E, we show that every such P admits an E-invariant Borel uniformization if and only if E is smooth. We also compute the definable complexity of counterexamples in the case where E is not smooth, using category, measure, and Ramsey-theoretic methods.

We provide two new proofs of a dichotomy of Miller classifying the pairs (E, P) such that P admits an E-invariant uniformization, for P with countable sections. In the process, we prove an $\aleph_0$-dimensional (G_0, H_0) dichotomy, generalizing dichotomies of Miller and Lecomte. We also show that the set of pairs (E, P) such that P has "large" sections and admits an E-invariant Borel uniformization is $\mathbf{\Sigma^1_2}$-complete; in particular, there is no analog of Miller's dichotomy for P with "large" sections.

Finally, we consider a less strict notion of invariant uniformization, where we select a countable nonempty subset of each section instead of a single point.

1 Introduction

1.1 Invariant uniformization and smoothness

Given sets X, Y and $P \subseteq X \times Y$ with $\mathrm{proj}_X(P) = X$, a **uniformization** of P is a function $f\colon X \to Y$ such that $\forall x \in X((x, f(x)) \in P)$. If now E is an equivalence relation on X, we say that P is ***E*-invariant** if $x_1 E x_2 \implies P_{x_1} = P_{x_2}$, where $P_x = \{y\colon (x,y) \in P\}$ is the x-**section** of P. Equivalently this means that P is invariant under the equivalence relation $E \times \Delta_Y$ on $X \times Y$, where Δ_Y is the equality relation on Y. In this case an ***E*-invariant uniformization** is a uniformization f such that $x_1 E x_2 \implies f(x_1) = f(x_2)$.

Also if E, F are equivalence relations on sets X, Y, resp., a **homomorphism** of E to F is a function $f\colon X \to Y$ such that $x_1 E x_2 \implies f(x_1) F f(x_2)$. Thus an invariant uniformization is a uniformization that is a homomorphism of E to Δ_Y.

Consider now the situation where X, Y are Polish spaces and P is a Borel subset of $X \times Y$. In this case standard results in descriptive set theory provide conditions which imply the existence of Borel uniformizations. These fall mainly into two categories, see [Kec95, Section 18]: "small section" and "large section" uniformization results. We will concentrate here on the following standard instances of these results:

Theorem 1.1 (Measure uniformization)**.** *Let X, Y be Polish spaces, μ a probability Borel measure on Y and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(\mu(P_x) > 0)$. Then P admits a Borel uniformization.*

Theorem 1.2 (Category uniformization)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(P_x$ is non-meager$)$. Then P admits a Borel uniformization.*

Theorem 1.3 (K_σ uniformization)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(P_x$ is non-empty and $K_\sigma)$. Then P admits a Borel uniformization.*

A special case of Theorem 1.3 is the following:

Theorem 1.4 (Countable uniformization)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(P_x$ is non empty and countable$)$. Then P admits a Borel uniformization.*

Suppose now that E is a Borel equivalence relation on X and P in any one of these results is E-invariant. When does there exist a **Borel E-invariant uniformization**, i.e., a Borel uniformization that is also a homomorphism of E to Δ_Y? We say that E satisfies **measure (resp., category, K_σ, countable) invariant uniformization** if for every Y, μ, P as in the corresponding uniformization theorem above, if P is moreover E-invariant, then it admits a Borel E-invariant uniformization.

The following gives a complete answer to this question. Recall that a Borel equivalence relation E on X is **smooth** if there is a Polish space Z and a Borel function $S\colon X \to Z$ such that $x_1 E x_2 \iff S(x_1) = S(x_2)$.

Theorem 1.5. *Let E be a Borel equivalence relation on a Polish space X. Then the following are equivalent:*

(i) E is smooth;

(ii) E satisfies measure invariant uniformization;

(iii) E satisfies category invariant uniformization;

(iv) E satisfies K_σ invariant uniformization;

(v) E satisfies countable invariant uniformization.

One can compute the exact definable complexity of counterexamples to invariant uniformization. Let E_0 denote the non-smooth Borel equivalence relation on $2^{\mathbb{N}}$ given by $x E_0 y \iff \exists m \forall n \geq m (x_n = y_n)$. In the proof of Theorem 1.5, it is shown that for $E = E_0$ on $X = 2^{\mathbb{N}}$ we have the following:

(1) Failure of measure invariant uniformization: There are Y, μ, E-invariant $P \in F_\sigma$ with $\mu(P_x) = 1$, for all $x \in X$, which has no Borel E-invariant uniformization.

(2) Failure of category invariant uniformization: There is Y and an E-invariant $Q \in G_\delta$ with Q_x comeager, for all $x \in X$, which has no Borel E-invariant uniformization.

(3) Failure of countable invariant uniformization: There is Y and an E-invariant $P \in F_\sigma$ such that P_x is non-empty and countable, for all $x \in X$, which has no Borel E-invariant uniformization.

The definable complexity of Q, P in (2), (3) is optimal. In the case of measure invariant uniformization, however, there are counterexamples which are G_δ, and this together with (1) gives the optimal definable complexity of counterexamples to measure invariant uniformization. These results are the contents of Theorems 1.6 and 1.7.

Theorem 1.6. *Let $X \subseteq 2^{\mathbb{N}}$ be the sequences with infinitely many ones. There is a Polish space Y, a probability Borel measure μ on Y and an E_0-invariant G_δ set $P \subseteq X \times Y$ with P_x comeager and $\mu(P_x) = 1$, for all $x \in X$, which has no Borel E_0-invariant uniformization.*

Theorem 1.7. *Let X, Y be Polish spaces, E a Borel equivalence relation on X and $P \subseteq X \times Y$ an E-invariant Borel relation. Suppose one of the following holds:*

(i) $P_x \in \mathbf{\Delta}^0_2$ and $\mu_x(P_x) > 0$, for all $x \in X$, and some Borel assignment $x \mapsto \mu_x$ of probability Borel measures μ_x on Y;

(ii) $P_x \in F_\sigma$ and P_x non-meager, for all $x \in X$;

(iii) $P_x \in G_\delta$ and P_x non-empty and K_σ (in particular countable), for all $x \in X$.

Then there is a Borel E-invariant uniformization.

The proof of Theorem 1.6 uses the Ramsey property.

1.2 Local dichotomies

The equivalence of (i) and (v) in Theorem 1.5 essentially reduces to the fact that if E is a countable Borel equivalence relation (i.e., one for which all of its equivalence classes are countable) which is not smooth, then the relation

$$(x, y) \in P \iff xEy,$$

is clearly E-invariant with countable nonempty sections but has no E-invariant uniformization. Considering the problem of invariant uniformization "locally", Miller [Milc] recently proved the following dichotomy that shows that this is essentially the only obstruction to (v). Below $E_0 \times I_{\mathbb{N}}$ is the equivalence relation on $2^{\mathbb{N}} \times \mathbb{N}$ given by $(x, m)E_0 \times I_{\mathbb{N}}(y, n) \iff xE_0y$. Also if E, F are equivalence relations on spaces X, Y, resp., an **embedding** of E into F is an injection $\pi \colon X \to Y$ such that $x_2Ex_2 \iff \pi(x_1)F\pi(x_2)$.

Theorem 1.8 ([Milc, Theorem 2])**.** *Let X, Y be Polish spaces, E a Borel equivalence relation on X and $P \subseteq X \times Y$ an E-invariant Borel relation with countable non-empty sections. Then exactly of the following holds:*

(1) There is a Borel E-invariant uniformization,

(2) There is a continuous embedding $\pi_X\colon 2^{\mathbb{N}} \times \mathbb{N} \to X$ of $E_0 \times I_{\mathbb{N}}$ into E and a continuous injection $\pi_Y\colon 2^{\mathbb{N}} \times \mathbb{N} \to Y$ such that for all $x, x' \in 2^{\mathbb{N}} \times \mathbb{N}$,

$$\neg(x \; E_0 \times I_{\mathbb{N}} \; x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset$$

and

$$P_{\pi_X(x)} = \pi_Y([x]_{E_0 \times I_{\mathbb{N}}}).$$

We provide a different proof of this dichotomy, using Miller's (G_0, H_0) dichotomy [Mil12] and Lecomte's $\aleph_0$-dimensional hypergraph dichotomy [Lec09]. Our proof relies on the following strengthening of $(i) \implies (v)$ of Theorem 1.5, which is interesting in its own right:

Theorem 1.9. *Let F be a smooth Borel equivalence relation on a Polish space X, Y be a Polish space, and $P \subseteq X \times Y$ be a Borel set with countable sections. Suppose that*

$$\bigcap_{x \in C} P_x \neq \emptyset$$

for every F-class C. Then P admits a Borel F-invariant uniformization.

We also prove an $\aleph_0$-dimensional (G_0, H_0)-type dichotomy, which generalizes Lecomte's dichotomy in the same way that the (G_0, H_0) dichotomy generalizes the G_0 dichotomy, and use this to give still another proof of Theorem 1.8.

In the case of countable uniformization, the Lusin-Novikov theorem asserts that P can be covered by the graphs of countably-many Borel functions. When E is smooth, the proof of Theorem 1.5 gives an invariant analogue of this fact (cf. Theorem 2.2). De Rancourt and Miller [dRM21] have shown that E_0 is essentially the only obstruction to invariant Lusin-Novikov:

Theorem 1.10 ([dRM21, Theorem 4.11])**.** *Let X, Y be Polish spaces, E a Borel equivalence relation on X and $P \subseteq X \times Y$ an E-invariant Borel relation with countable non-empty sections. Then exactly one of the following holds:*

(1) There is a sequence $g_n : X \to Y$ of Borel E-invariant uniformizations with $P = \bigcup_n \operatorname{graph}(g_n)$,

(2) There is a continuous embedding $\pi_X : 2^{\mathbb{N}} \to X$ of E_0 into E and a continuous injection $\pi_Y : 2^{\mathbb{N}} \to Y$ such that for all $x \in 2^{\mathbb{N}}$, $P(\pi_X(x), \pi_Y(x))$.

We provide a different proof of this theorem in Section 4.4, directly from Miller's (G_0, H_0) dichotomy.

1.3 Anti-dichotomy results

Our next result can be viewed as a sort of anti-dichotomy theorem for large-section invariant uniformizations (see also the discussion in [TV21, Section 1]). Informally, dichotomies such as Theorem 1.8 provide upper bounds on the complexity of the collection of Borel sets satisfying certain combinatorial properties. Thus, one method of showing that there is no analogous dichotomy is to provide lower bounds on the complexity of such sets.

In order to state this precisely, we first fix a "nice" parametrization of the Borel relations on $\mathbb{N}^\mathbb{N}$, i.e., a $\mathbf{\Pi}^1_1$ set $D \subseteq 2^\mathbb{N}$ and a map $D \ni d \mapsto D_d$ such that each $D_d \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}, d \in D$ is Borel, each Borel set in $\mathbb{N}^\mathbb{N}$ appears as some D_d, and so that these satisfy some natural definability properties (cf. [AK00, Section 5]).

Define now

$$\mathcal{P} = \{(d, e) : D_d \text{ is an equivalence relation on } \mathbb{N}^\mathbb{N} \text{ and } D_e \text{ is } D_d\text{-invariant}\},$$

and let $\mathcal{P}^{unif}$ denote the set of pairs $(d, e) \in \mathcal{P}$ for which D_e admits a D_d-invariant uniformization. More generally, for any set A of properties of sets $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$, let $\mathcal{P}_A$ (resp. $\mathcal{P}^{unif}_A$) denote the set of pairs (d, e) in $\mathcal{P}$ (resp. $\mathcal{P}^{unif}$) such that D_e satisfies all of the properties in A. Let $\mathcal{P}_{ctble}$ (resp. $\mathcal{P}^{unif}_{ctble}$) denote $\mathcal{P}_A$ (resp. $\mathcal{P}^{unif}_A$) for A consisting of the property that P has countable sections.

One can easily check that $\mathcal{P}$ is $\mathbf{\Pi}^1_1$ and that $\mathcal{P}^{unif}$ is $\mathbf{\Sigma}^1_2$. The same is true for $\mathcal{P}_{ctble}$ and $\mathcal{P}^{unif}_{ctble}$. In the latter case, however, the effective version of Theorem 1.8 (see Theorem 4.14) gives a better bound on the complexity:

Proposition 1.11. *The set $\mathcal{P}^{unif}_{ctble}$ is $\mathbf{\Pi}^1_1$.*

By contrast, in the case of large sections, we prove the following, where a set B in a Polish space X is called $\mathbf{\Sigma}^1_2$-complete if it is $\mathbf{\Sigma}^1_2$, and for all zero-dimensional Polish spaces Y and $\mathbf{\Sigma}^1_2$ sets $C \subseteq Y$ there is a continuous function $f : Y \to X$ such that $C = f^{-1}(B)$.

Theorem 1.12. *The set $\mathcal{P}^{unif}_A$ is $\mathbf{\Sigma}^1_2$-complete, where A is one of the following sets of properties of $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$:*

1. *P has non-meager sections;*
2. *P has non-meager G_δ sections;*

3. *P has non-meager sections and is G_δ;*

4. *P has μ-positive sections for some probability Borel measure μ on $\mathbb{N}^\mathbb{N}$;*

5. *P has μ-positive F_σ sections for some probability Borel measure μ on $\mathbb{N}^\mathbb{N}$;*

6. *P has μ-positive sections for some probability Borel measure μ on $\mathbb{N}^\mathbb{N}$ and is F_σ.*

The same holds for comeager instead of non-meager, and μ-conull instead of μ-positive.

In fact, there is a hyperfinite Borel equivalence relation E with code $d \in D$ such that for all such A above, the set of $e \in D$ such that $(d, e) \in \mathcal{P}_A^{unif}$ is $\boldsymbol{\Sigma}_2^1$-complete.

Problem 1.13. *Is there an analogous dichotomy or anti-dichotomy result for the case where P has K_σ sections?*

While we do not know the answer to this problem, we note that Theorem 1.9 is false when the sections are only assumed to be K_σ:

Proposition 1.14. *There is a smooth countable Borel equivalence relation F on $\mathbb{N}^\mathbb{N}$ and an open set $P \subseteq \mathbb{N}^\mathbb{N} \times 2^\mathbb{N}$ such that*

$$\bigcap_{x \in C} P_x \neq \emptyset$$

for every F-class C, but which does not admit a Borel F-invariant uniformization.

1.4 Invariant countable uniformization

We next consider a somewhat less strict notion of invariant uniformization, where instead of selecting a single point in each section we select a countable nonempty subset. More precisely, given Polish spaces X, Y, a Borel equivalence relation E on X and an E-invariant Borel set $P \subseteq X \times Y$, with $\mathrm{proj}_X(P) = X$, a Borel ***E*-invariant countable uniformization** is a Borel function $f\colon X \to Y^\mathbb{N}$ such that $\forall x \in X \forall n \in \mathbb{N}((x, f(x)_n) \in P)$ and

$x_1 E x_2 \implies \{f(x_1)_n \colon n \in \mathbb{N}\} = \{f(x_2)_n \colon n \in \mathbb{N}\}$. Equivalently, if for each Polish space Y, we denote by E^Y_{ctble} the equivalence relation on $Y^{\mathbb{N}}$ given by

$$(x_n) E^Y_{ctble} (y_n) \iff \{x_n \colon n \in \mathbb{N}\} = \{y_n \colon n \in \mathbb{N}\},$$

then an E-invariant countable uniformization is a Borel homomorphism f of E to E^Y_{ctble} such that for each x, n, we have that $(x, f(x)_n) \in P$.

We say that E satisfies **measure (resp., category, $\boldsymbol{K_\sigma}$) countable invariant uniformization** if for every Y, μ, P as in the corresponding uniformization theorem above, if P is moreover E-invariant, then it admits a Borel E-invariant countable uniformization.

Recall that a Borel equivalence relation E on X is **reducible to countable** if there is a Polish space Z, a countable Borel equivalence relation F on Z and a Borel function $S \colon X \to Z$ such that $x_1 E x_2 \iff S(x_1) F S(x_2)$.

As in the proof below of Theorem 1.5, part **(A)**, one can see that if a Borel equivalence relation E on X is reducible to countable, then E satisfies measure (resp. category, K_σ) countable invariant uniformization. We conjecture the following:

Conjecture 1.15. *Let E be a Borel equivalence relation on a Polish space X. Then the following are equivalent:*

(a) E is reducible to countable;

(b) E satisfies measure countable invariant uniformization;

(c) E satisfies category countable invariant uniformization;

(d) E satisfies K_σ countable invariant uniformization.

We discuss some partial results in Section 5.

1.5 Further invariant uniformization results and smoothness

We have so far considered the existence of Borel invariant uniformizations, generalizing the standard "small section" and "large section" uniformization theorems. One can also consider invariant analogues of uniformization theorems for more general pointclasses, such as the following:

Theorem 1.16 (Jankov, von Neumann uniformization [Kec95, 18.1])**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ be a $\boldsymbol{\Sigma^1_1}$ set such that P_x is nonempty, for all $x \in X$. Then P has a uniformization function which is $\sigma(\boldsymbol{\Sigma^1_1})$-measurable.*

Theorem 1.17 (Novikov-Kondô uniformization [Kec95, 36.14])**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ be a $\mathbf{\Pi}^1_1$ set such that P_x is non-empty, for all $x \in X$. Then P has a uniformizatoin function whose graph is $\mathbf{\Pi}^1_1$.*

Let E be a Borel equivalence relation on X. We say E satisfies **Jankov-von Neumann (resp. Novikov-Kondô) invariant uniformization** if for every Y, P as in the corresponding uniformization theorem above, if P is moreover E-invariant, then it admits an E-invariant uniformization which is definable in the same sense as in the corresponding uniformization theorem.

The following characterization of those Borel equivalence relations that satisfy these properties essentially follows from the proof of Theorem 1.5.

Theorem 1.18. *Let E be a Borel equivalence relation on a Polish space X. Then the following are equivalent:*
(i) E is smooth;
(ii) E satisfies Jankov-von Neumann invariant uniformization;
(iii) E satisfies Novikov-Kondô invariant uniformization.

1.6 Remarks on invariant uniformization over products

One can consider more generally the question of invariant uniformization over products. Let X, Y be Polish spaces, E a Borel equivalence on X, F a Borel equivalence on Y, and $P \subseteq X \times Y$ an $E \times F$-invariant set. In this case, one can ask whether there is an $E \times F$-invariant Borel set $U \subseteq P$ so that each section U_x intersects one, or even finitely-many, F-classes. This paper then considers the special case where $F = \Delta_Y$ is equality.

In the case where P has countable sections and F is smooth, one can reduce this to the case where F is equality to get analogues of Theorems 1.8 and 1.10.

Miller [Milc, Theorem 2.1] has proved a generalization of Theorem 1.8 where P has countable sections and the equivalence classes of F are countable, and de Rancourt and Miller [dRM21, Theorem 4.11] have proved a generalization of Theorem 1.10 where the sections of P are contained in countably many F-classes (but are not necessarily countable).

The problem of invariant uniformization is also discussed in [Mye76; BM75] where they consider the question of invariant uniformization over products when E, F come from Polish group actions, and specifically when

E, F are the isomorphism relation on a class of structures. Myers [Mye76, Theorem 10] gives an example in which there is no Baire-measurable invariant uniformization, so that in particular the invariant Jankov-von Neumann and invariant Novikov-Kondô uniformization don't hold.

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2 Proof of Theorem 1.5

(A) We first show that (i) implies (ii), the proof that (i) implies (iii) being similar. Fix a Polish space Z and a Borel function $S\colon X \to Z$ such that $x_1 E x_2 \iff S(x_1) = S(x_2)$. Fix also Y, μ, P as in the definition of measure invariant uniformization. Define $P^* \subseteq Z \times Y$ as follows:

$$(z, y) \in P^* \iff \forall x \in X \big(S(x) = z \implies (x, y) \in P\big).$$

Then P^* is $\mathbf{\Pi}^1_1$ and we have that

$$S(x) = z \implies P^*_z = P_x,$$
$$z \notin S(X) \implies P^*_z = Y.$$

Thus $\forall z \in Z(\mu(P^*_z) > 0)$. Then, by [Kec95, 36.24], there is a Borel function $f^*\colon Z \to Y$ such that $\forall z \in Z((z, f^*(z)) \in P^*)$. Put

$$f(x) = f^*(S(x)).$$

Then f is an E-invariant uniformization of P.

We next prove that (i) implies (iv) (and therefore (v)). Fix Z, S as in the previous case and Y, P as in the definition of K_σ invariant uniformization. Define P^* as before. Then $A = \{(z, y)\colon \exists x \in X(S(x) = z \ \& \ P(x, y))\}$ is a $\mathbf{\Sigma}^1_1$ subset of P^*, so by the Lusin separation theorem there is a Borel subset P^{**} of P^* such that $A \subseteq P^{**}$. By [Kec95, 35.47], the set C of all $z \in Z$ such that P^{**}_z is K_σ is $\mathbf{\Pi}^1_1$ and contains the $\mathbf{\Sigma}^1_1$ set $S(X)$, so by separation there is a Borel set B with $A \subseteq B \subseteq C$. Then if $Q \subseteq Z \times Y$ is defined by

$$(z, y) \in Q \iff z \in B \ \& \ (z, y) \in P^{**},$$

we have that

$$S(x) = z \implies Q_z = P_x,$$

and every Q_z is K_σ. It follows, by [Kec95, 35.46], that $D = \mathrm{proj}_Z(Q)$ is Borel and there is a Borel function $g\colon D \to Y$ such that $\forall z \in D(z, g(z)) \in Q$. Since $f(X) \subseteq D$, the function

$$f(x) = g(S(x)).$$

is an E-invariant uniformization of P.

(B) We will next show that $\neg$(i) implies $\neg$(ii), $\neg$(iii) and $\neg$(v) (and thus also $\neg$(iv)). We will use the following lemma. Below for Borel equivalence relations E, E' on Polish spaces X, X', resp., we write $E \leq_B E'$ iff there is a Borel map $f\colon X \to X'$ such that $x_1 E x_2 \iff f(x_1) E' f(x_2)$, i.e., E can be **Borel reduced** to E' (via the reduction f).

Lemma 2.1. *Let E, E' be Borel equivalence relations on Polish spaces X, X', resp., such that $E \leq_B E'$. If E fails (ii) (resp., (iii), (iv), (v)), so does E'.*

Proof. Let $f\colon X \to X'$ be a Borel reduction of E into E'. Assume first that E fails (ii) with witness Y, μ, P. Define $P' \subseteq X' \times Y$ by

$$(x', y) \in P' \iff \forall x \in X\Big(f(x) E' x' \implies (x, y) \in P\Big).$$

Then note that

$$f(x) E' x' \implies P'_{x'} = P_x,$$
$$x' \notin [f(X)]_{E'} \implies P'_{x'} = Y.$$

Now clearly P' is $\mathbf{\Pi}^1_1$ and invariant under the Borel equivalence relation $E' \times \Delta_Y$. Then by a result of Solovay (see [Kec95, 34.6]), there is a $\mathbf{\Pi}^1_1$-rank $\varphi\colon P' \to \omega_1$ which is $E' \times \Delta_Y$-invariant. Consider then the $\mathbf{\Sigma}^1_1$ subset P'' of P' defined by:

$$(x', y) \in P'' \iff \exists x \in X\Big(f(x) E' x' \;\&\; (x, y) \in P\Big).$$

By boundedness there is a Borel $E' \times \Delta_Y$-invariant set P''' with $P'' \subseteq P''' \subseteq P'$. Let now $Z \subseteq X'$ be defined by

$$x' \in Z \iff \mu(P'''_{x'}) > 0.$$

Then Z is Borel and E'-invariant and contains $[f(X)]_{E'}$. Finally define $Q \subseteq X' \times Y$ by

$$(x', y) \in Q \iff \big(x' \in Z \;\&\; (x', y) \in P'''\big) \text{ or } x' \notin Z.$$

Then $f(x) = x' \implies Q_{x'} = P_x$, so Y, μ, Q witnesses the failure of (ii) for E'.

The case of (iii) is similar and we next consider the case of (iv). Repeat then the previous argument for case (ii) until the definition of P'''. Then define $Z' \subseteq X'$ by

$$x' \in Z' \iff P'''_{x'} \text{ is } K_\sigma \text{ and nonempty.}$$

Then Z' is $\mathbf{\Pi^1_1}$, by [Kec95, 35.47] and the relativization of the fact that every nonempty Δ^1_1 K_σ set contains a Δ^1_1 member, see [Mos09, 4F.15]. It is also E'-invariant and contains $[f(X)]_{E'}$. Let then Z be E'-invariant Borel with $[f(X)]_{E'} \subseteq Z \subseteq Z'$ and define Q as before but replacing "$x' \notin Z$" by "$(x' \notin Z$ and $y = y_0)$", for some fixed $y_0 \in Y$. Then Y, Q witnesses the failure of (iv) for E'.

Finally, the case of (v) is similar to (iv) by now defining

$$x' \in Z' \iff P'''_{x'} \text{ is countable and nonempty.}$$

and using that Z' is $\mathbf{\Pi^1_1}$ by [Kec95, 35.38] (and [Mos09, 4F.15] again). $\square$

Assume now that E is not smooth. Then by [HKL90] we have $E_0 \leq_B E$. Thus by Lemma 2.1 it is enough to show that E_0 fails (ii), (iii), and (v) (thus also (iv)).

We first prove that E_0 fails (ii). We view here $2^{\mathbb{N}}$ as the Cantor group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ with pointwise addition $+$ and we let μ be the Haar measure, i.e., the usual product measure. Let then $A \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ be an F_σ set which has μ-measure 1 but is meager. Let $X = Y = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ and define $P \subseteq X \times Y$ as follows:

$$(x, y) \in P \iff \exists x' E_0 x (x' + y \in A).$$

Clearly P is F_σ and, since $P_x = \bigcup_{x' E_0 x}(A - x')$, clearly $\mu(P_x) = 1$. Moreover P is E_0-invariant. Assume then, towards a contradiction that f is a Borel E_0-invariant uniformization. Since $x E_0 x' \implies f(x) = f(x')$, by generic ergodicity of E_0 there is a comeager Borel E_0-invariant set $C \subseteq X$ and y_0 such that $\forall x \in C (f(x) = y_0)$, thus $\forall x \in C (x, y_0) \in P$, so $\forall x \in C \exists x' E_0 x (x' \in A - y_0)$. If $G \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is the subgroup consisting of the eventually 0 sequences, then $xE_0y \iff \exists g \in G(g+x=y)$, thus $C = \bigcup_{g\in G}(g+(A-y_0))$, so C is meager, a contradiction.

To show that E_0 fails (v), define

$$(x,y) \in P \iff xE_0y.$$

Then any Borel E_0-invariant uniformization of P gives a Borel selector for E_0, a contradiction.

Finally to see that E_0 fails (iii), use above $B = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \setminus A$, instead of A, to produce a G_δ set Q as follows:

$$(x,y) \in Q \iff \forall x' E_0 x(x'+y \in B).$$

Then Q is E_0-invariant and has comeager sections. If g is a Borel E_0-invariant uniformization, then by the ergodicity of E_0, there is a μ-measure 1 set D and y_0 such that $\forall x \in D \forall x' E_0 x(x' \in B - y_0)$, so $D \subseteq B - y_0$, thus $\mu(D) = 0$, a contradiction.

This completes the proof of Theorem 1.5.

(C) We note the following strengthening of Theorem 1.5 in the case that E is smooth, where $K(Y)$ denotes the Polish space of compact subsets of Y [Kec95, 4.F]:

Theorem 2.2. *Let X, Y be Polish spaces, E be a smooth Borel equivalence relation on X, and $P \subseteq X \times Y$ be a Borel E-invariant set with non-empty sections.*

1. *If P has countable sections, then $P = \bigcup_n \mathrm{graph}(g_n)$ for a sequence of E-invariant Borel maps $g_n : X \to Y$.*

2. *If P has K_σ sections, then $P_x = \bigcup_n K_n(x)$ for a sequence of E-invariant Borel maps $K_n : X \to K(Y)$.*

3. *If P has comeager sections, then $P \supseteq \bigcap_n U_n$ for a sequence of E-invariant Borel sets $U_n \subseteq X \times Y$ with dense open sections. Moreover, if P has dense G_δ sections, we can find such U_n with $P = \bigcap_n U_n$.*

Proof. The first two assertions follow from [Kec95, 18.10, 35.46] applied to Q from the proof of (i) $\implies$ (iv) of Theorem 1.5.

For the third, let Z, S, P^*, P^{**} be as in the proof of (i) $\implies$ (iv). By [Kec95, 16.1] the set C of all z for which P^{**}_z is comeager is Borel, so

$Q(z,y) \iff [C(z) \implies P^{**}(z,y)]$ is Borel with comeager sections and $S(x) = z \implies P_x = Q_z$.

If moreover P has G_δ sections, we instead let A be the set of all z for which P^{**}_z is comeager and G_δ, which is $\boldsymbol{\Pi}^1_1$ by [Kec95, 35.47]. Then $S(X) \subseteq A$ is $\boldsymbol{\Sigma}^1_1$, so by the Lusin separation theorem there is a Borel set $S(X) \subseteq C \subseteq A$. We then define Q as above, so that Q moreover has G_δ sections.

The result then follows by [Kec95, 35.43]. □

(D) Theorems 1.1–1.3 are effective, meaning that whenever P is (lightface) Δ^1_1 and satisfies the hypotheses of one of these theorems, then P admits a Δ^1_1 uniformization (cf. [Mos09, 4F.16, 4F.20] and the discussion afterwards). Similarly, [HKL90] implies that if E is smooth and Δ^1_1 then it has a Δ^1_1 reduction to $\Delta(2^{\mathbb{N}})$. The proof of Theorem 1.5 therefore gives the following effective refinement:

Theorem 2.3. *Let E be a smooth Δ^1_1 equivalence relation on $\mathbb{N}^{\mathbb{N}}$ and $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be Δ^1_1 and E-invariant. Then P admits a Δ^1_1 E-invariant uniformization whenever one of the following holds:*

(i) P has μ-positive sections, for some Δ^1_1 probability measure μ on $\mathbb{N}^{\mathbb{N}}$;

(ii) P has non-meager sections;

(iii) P has non-empty K_σ sections;

(iv) P has non-empty countable sections.

In (i) above, we identify probability Borel measures on $\mathbb{N}^{\mathbb{N}}$ with points in $[0,1]^{\mathbb{N}^{<\mathbb{N}}}$ [Kec95, 17.7].

It is also interesting to consider whether the converse holds. For example, let E be a Δ^1_1 equivalence relation on $\mathbb{N}^{\mathbb{N}}$, and suppose that for every Δ^1_1 E-invariant set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ which satisfies one of (i)-(iv) above, P admits a Δ^1_1 E-invariant uniformization. Must it be the case that E is smooth?

If we replace Δ^1_1 by Borel, then E must indeed be smooth by Theorem 1.5. However, to prove this we use the fact that every non-smooth Δ^1_1 equivalence relation embeds E_0 [HKL90], and this is not effective: There are non-smooth Δ^1_1 equivalence relations on $\mathbb{N}^{\mathbb{N}}$ which do not admit Δ^1_1 embeddings of E_0.

Restricting our attention to those P which have countable sections, it turns out that the converse to Theorem 2.3 is false. In fact, using the theory of turbulence, one can construct the following very strong counterexample:

Theorem 2.4. *There is a Π^0_1 set $N \subseteq \mathbb{N}^{\mathbb{N}}$ and a Δ^1_1 equivalence relation E on N which is not smooth, and such that every Δ^1_1 E-invariant set $P \subseteq N \times \mathbb{N}^{\mathbb{N}}$*

with non-empty countable sections is invariant, meaning that $P_x = P_{x'}$ for all $x, x' \in N$.

Corollary 2.5. *There is a Δ^1_1 equivalence relation F on $\mathbb{N}^\mathbb{N}$ which is not smooth, and such that every Δ^1_1 F-invariant set $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ with non-empty countable sections admits a Δ^1_1 F-invariant uniformization.*

Proof. Let N, E be as in Theorem 2.4 and define

$$xFx' \iff (x = x') \vee (x, x' \in N \ \ \& \ \ xEx').$$

Suppose now that $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ were Δ^1_1 and F-invariant. Let $y \in A \iff \exists x \in N(P(x, y)) \iff \forall x \in N(P(x, y))$. Then A is countable and Δ^1_1, and $P_x = A$ for all $x \in N$. In particular, A contains a Δ^1_1 point, say y_0.

By the effective Lusin-Novikov theorem, there is a Δ^1_1 uniformization f of P. Letting $g(x) = f(x)$ for $x \notin N$, and $g(x) = y_0$ otherwise, gives a Δ^1_1 F-invariant uniformization of P. □

Proof of Theorem 2.4. Consider the group $\mathbb{R}^\mathbb{N}$ and the translation action of $\ell^1 \subseteq \mathbb{R}^\mathbb{N}$ on $\mathbb{R}^\mathbb{N}$, which is turbulent by [Kec03, Section 10(ii)]. Let F be the induced equivalence relation, which is clearly Δ^1_1.

Let $\mathbb{F} \subseteq \mathbb{N}$ be the Π^1_1 set of codes for the Δ^1_1 functions from $\mathbb{R}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$, and for $n \in \mathbb{F}$ let f_n be the function that it codes. Let also $\mathbb{H} \subseteq \mathbb{F}$ be the Π^1_1 set of those n for which f_n is a homomorphism of F into $E^{\mathbb{N}^\mathbb{N}}_{ctble}$.

Finally, let $\mathbb{D} \subseteq \mathbb{N}$ denote the usual Π^1_1 set of codes for the Δ^1_1 subsets of $\mathbb{R}^\mathbb{N}$, and for $n \in \mathbb{D}$ let $\mathbb{D}_n$ be the set that it codes.

By the proof of [Kec03, Theorem 12.5(i) $\implies$ (ii)], for each $n \in \mathbb{H}$ there is Δ^1_1 comeager F-invariant set $C_n \subseteq \mathbb{R}^\mathbb{N}$ which f_n maps to a single $E^{\mathbb{N}^\mathbb{N}}_{ctble}$-class. Moreover, there is a computable map $n \mapsto n^*$ such that if $n \in \mathbb{H}$ then $n^* \in \mathbb{D}$ and $C_n = \mathbb{D}_{n^*}$.

Put $C = \bigcap_{n \in \mathbb{H}} C_n \subseteq \mathbb{R}^\mathbb{N}$. Then C is comeager, F-invariant and Σ^1_1, since

$$a \in C \iff \forall n (n \in \mathbb{H} \implies a \in \mathbb{D}_{n^*}).$$

Moreover, for each Δ^1_1 homomorphism f of F to $E^{\mathbb{N}^\mathbb{N}}_{ctble}$, $f \restriction C$ maps into a single $E^{\mathbb{N}^\mathbb{N}}_{ctble}$-class.

Let now $N \subseteq \mathbb{N}^\mathbb{N}$ be Π^0_1 and $c : N \to \mathbb{R}^\mathbb{N}$ be a Δ^1_1 map such that $c(N) = C$. Define the Δ^1_1 equivalence relation E on N by

$$xEx' \iff c(x)Fc(x').$$

We will show that this E works.

Let $P \subseteq N \times \mathbb{N}^{\mathbb{N}}$ be E-invariant with non-empty countable sections. Define $Q \subseteq C \times \mathbb{N}^{\mathbb{N}}$ by

$$\begin{aligned}(a,y) \in Q &\iff a \in C \ \& \ \exists x \in N(c(x) = a \ \& \ P(x,y)) \\ &\iff a \in C \ \& \ \forall x \in N(c(x) = a \implies P(x,y)).\end{aligned}$$

Note that Q is F-invariant. Moreover, Q is Δ^1_1 on the Σ^1_1 set $C \times \mathbb{N}^{\mathbb{N}}$, i.e., it is the intersection of $C \times \mathbb{N}^{\mathbb{N}}$ with a Σ^1_1 set in $\mathbb{R}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ as well as with a Π^1_1 set in $\mathbb{R}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. By Σ^1_1 separation, there is a Δ^1_1 set $R \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $R \cap (C \times \mathbb{N}^{\mathbb{N}}) = Q$.

Let now $C^* \subseteq \mathbb{R}^{\mathbb{N}}$ be defined by

$$a \in C^* \iff \forall a'[aFa' \implies R_a = R_{a'} \ \& \ R_a \text{ is countable and non-empty}].$$

Then C^* is Π^1_1, F-invariant and contains C, so there is a Δ^1_1 set B which is F-invariant and such that $C \subseteq B \subseteq C^*$. Finally, define $S \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ by

$$(a,y) \in S \iff [a \in B \ \& \ R(a,y)] \vee [a \notin B \ \& \ y = y_0]$$

for some fixed Δ^1_1 point y_0 in $\mathbb{N}^{\mathbb{N}}$. Then S is Δ^1_1, F-invariant, and has non-empty countable sections.

Let $s : \mathbb{R}^{\mathbb{N}} \to (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ be a Δ^1_1 homomorphism of F to $E_{ctble}^{\mathbb{N}^{\mathbb{N}}}$ for which $S_a = \{s(a)_n\}$ for all $a \in \mathbb{R}^{\mathbb{N}}$, which exists by the effective Lusin-Novikov theorem. By the definition of C, we have that $s \restriction C$ maps into a single $E_{ctble}^{\mathbb{N}^{\mathbb{N}}}$-class. Let A be the corresponding countable set. Then for $a \in C$ and any $x \in N$ with $c(x) = a$,

$$A = S_a = R_a = Q_a = P_x,$$

so $P_x = A$ for all $x \in N$.

It remains to check that E is not smooth. To see this, note that $F \restriction C$ has at least two classes (as every F-class is meager), and hence so does E. If E were smooth, then there would be a Δ^1_1 map $f : N \to \mathbb{N}^{\mathbb{N}}$ for which

$$xEx' \iff f(x) = f(x').$$

But then graph(f) would be Δ^1_1, E-invariant, have non-empty countable sections, and satisfy $P_x \neq P_{x'}$ for some $x, x' \in N$, a contradiction. $\square$

Problem 2.6. *Is there a Δ^1_1 equivalence relation E on $\mathbb{N}^\mathbb{N}$ which is not smooth, and such that all Δ^1_1 E-invariant sets $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ satisfying one of (i)-(iii) in Theorem 2.3 admit a Δ^1_1 E-invariant uniformization?*

Finally, we remark that if E is a Δ^1_1 equivalence relation which is not smooth, then there is a continuous embedding of E_0 into E which is $\Delta^1_1(\mathcal{O})$. In particular, the converse of Theorem 2.3 holds if we consider all such $P \in \Delta^1_1(\mathcal{O})$.

3 Proofs of Theorems 1.6 and 1.7

(A) We first prove Theorem 1.7.

Let $F(Y)$ denote the Effros Borel space of closed subsets of Y (cf. [Kec95, 12.C]). Suppose $P_x \in F_\sigma$, for all $x \in X$, and that there is an E-invariant Borel map $x \mapsto F_x \in F(Y)$ such that P_x is non-meager in F_x for all $x \in X$. By [Kec95, 12.13], there is a sequence of E-invariant Borel functions $y_n : X \to Y$ such that $\{y_n(x)\}$ is dense in F_x for all $x \in X$. Since P_x is non-meager and F_σ in F_x, P_x contains an open set in F_x, and in particular contains some $y_n(x)$. Thus the map taking x to the least $y_n(x)$ such that $P(x, y_n(x))$ is an E-invariant Borel uniformization of P.

It remains only to show that in each of the cases (i), (ii), (iii), such an assignment $x \mapsto F_x$ exists. In (ii), we can take $F_x = Y$.

Consider case (i), that there is a Borel assignment $x \mapsto \mu_x$ of probability Borel measures on Y such that $P_x \in \mathbf{\Delta^0_2}$ and $\mu_x(P_x) > 0$, for all $x \in X$. Let ν_x denote the probability Borel measure μ_x restricted to P_x, i.e., $\nu_x(A) = \mu_x(A \cap P_x)/\mu_x(P_x)$, and define F_x to be the support of ν_x, i.e., the smallest ν_x-conull closed set in Y.

Since F_x is the support of ν_x, any open set in F_x is ν_x-positive, and therefore any ν_x-null F_σ set in F_x is meager. Now P_x is G_δ and ν_x-conull in F_x, so P_x is comeager in F_x, for all $x \in X$. Thus it remains only to show that the map $x \mapsto F_x$ is Borel. To see this, we observe that

$$F_x \cap U \neq \emptyset \iff \nu_x(U) > 0 \iff \mu_x(U \cap P_x) > 0$$

is Borel, by [Kec95, 17.25].

Finally, consider case (iii), that $P_x \in G_\delta$ and P_x is non-empty and K_σ for all $x \in X$. Let F_x be the closure of P_x in Y. Then P_x is dense G_δ in F_x, so it remains to check that $x \mapsto F_x$ is Borel. Indeed,

$$F_x \cap U \neq \emptyset \iff P_x \cap U \neq \emptyset,$$

and this is Borel by the Arsenin-Kunugui theorem [Kec95, 18.18], as $P_x \cap U$ is K_σ for all $x \in X$.

(B) We now prove Theorem 1.6.

Let $X = [\mathbb{N}]^{\aleph_0}$ denote the space of infinite subsets of $\mathbb{N}$. By identifying subsets of $\mathbb{N}$ with their characteristic functions, we can view X as an E_0-invariant G_δ subspace of $2^{\mathbb{N}}$. Note that this is a dense G_δ in $2^{\mathbb{N}}$, and it is μ-conull, where μ is the uniform product measure on $2^{\mathbb{N}}$. We let E denote the equivalence relation E_0 restricted to X.

Let $Y = 2^{\mathbb{N}}$, and define $P \subseteq X \times Y$ by

$$P(A, B) \iff |A \setminus B| = |A \cap B| = \aleph_0.$$

Then P is G_δ and E-invariant, and P_x is comeager for all $x \in X$. By the Borel-Cantelli lemma, one easily sees that $\mu(P_x) = 1$ for all $x \in X$.

We claim that P does not admit an E-invariant Borel uniformization. Indeed, suppose such a uniformization $f : X \to Y$ existed. By [Kec95, 19.19], there is some $A \in X$ such that $f \restriction [A]^{\aleph_0}$ is continuous, where $[A]^{\aleph_0}$ denotes the space of infinite subsets of A. Since E-classes are dense in $[A]^{\aleph_0}$, $f \restriction [A]^{\aleph_0}$ is constant, say with value B. Then $f(A) = B$, so $P(A, B)$ and $A \cap B$ is infinite. But then $A \cap B \in [A]^{\aleph_0}$, so $f(A \cap B) = B$. But $(A \cap B) \setminus B$ is not infinite, so $\neg P(A \cap B, B)$, a contradiction.

Remark 3.1. *Using the same Ramsey-theoretic arguments, one can show that the following examples also do not admit E-invariant uniformizations:*

1. *Let Y be the space of graphs on $\mathbb{N}$ and set $Q(A, G)$ iff for all finite disjoint sets $x, y \subseteq \mathbb{N}$ there is some $a \in A$ which is adjacent (in G) to every element of x and no element of y, i.e., A contains witnesses that G is the random graph.*

2. *Let $Y = [\mathbb{N}]^{\aleph_0}$, and for $B \in Y$ let $f_B : \mathbb{N} \to \mathbb{N}$ denote its increasing enumeration. Then take $R(A, B)$ iff $f_B(A)$ contains infinitely many even and infinitely many odd elements.*

As with P above, Q, R both have μ-conull dense G_δ sections.

4 Dichotomies and anti-dichotomies

4.1 Proof of Theorem 1.8

Here we derive Miller's dichotomy Theorem 1.8 for sets with countable sections, from Miller's (G_0, H_0) dichotomy [Mil12] and Lecomte's $\aleph_0$-dimensional hypergraph dichotomy [Lec09].

We begin by noting the following equivalent formulations of the second alternative in Theorem 1.8.

Proposition 4.1. *Let X, Y be Polish spaces, E a Borel equivalence relation on X and $P \subseteq X \times Y$ an E-invariant Borel relation with countable non-empty sections. Then the following are equivalent:*

(2) There is a continuous embedding $\pi_X : 2^{\mathbb{N}} \times \mathbb{N} \to X$ of $E_0 \times I_{\mathbb{N}}$ into E and a continuous injection $\pi_Y : 2^{\mathbb{N}} \times \mathbb{N} \to Y$ such that for all $x, x' \in 2^{\mathbb{N}} \times \mathbb{N}$,

$$\neg(x \; E_0 \times I_{\mathbb{N}} \; x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset$$

and

$$P_{\pi_X(x)} = \pi_Y([x]_{E_0 \times I_{\mathbb{N}}}).$$

(3) There is a continuous embedding $\pi_X : 2^{\mathbb{N}} \to X$ of E_0 into E and a continuous injection $\pi_Y : 2^{\mathbb{N}} \to Y$ such that for all $x, x' \in 2^{\mathbb{N}}$,

$$\neg(x \; E_0 \; x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset$$

and

$$\pi_Y(x) \in P_{\pi_X(x)}.$$

(4) There is a continuous embedding $\pi_X : 2^{\mathbb{N}} \to X$ of E_0 into E such that for all $x, x' \in 2^{\mathbb{N}}$,

$$\neg(x \; E_0 \; x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset.$$

Proof. Clearly (2) $\implies$ (3) $\implies$ (4). Assume now that (4) holds, and is witnessed by π_X. Let g be a uniformization of P and $\pi_Y = g \circ \pi_X$. Since π_Y is countable-to-one, by the Lusin-Novikov theorem there is a Borel non-meager set $B \subseteq 2^{\mathbb{N}}$ on which π_Y is injective. We then recursively construct a continuous embedding of E_0 into $E_0 \restriction B$, and compose this with π_X, π_Y to get maps witnessing (3).

Now suppose (3) holds, and is witnessed by π_X, π_Y. Let h be a continuous embedding of $E_0 \times I_{\mathbb{N}}$ into E_0, and let $\tilde{\pi}_X = \pi_X \circ h$. Let F be the equivalence relation on Y defined by yFy' iff $y = y'$ or there is some $x \in 2^{\mathbb{N}} \times \mathbb{N}$ such that $P(\tilde{\pi}_X(x), y)$ and $P(\tilde{\pi}_X(x), y')$. If $y \neq y'$, then the set of x witnessing that yFy' is a single $E_0 \times I_{\mathbb{N}}$-class, so by Lusin-Novikov F is Borel. Thus, $\pi_Y \circ h$ is an embedding of $E_0 \times I_{\mathbb{N}}$ into the countable Borel equivalence relation F, and by compressibility we can turn this into an invariant Borel embedding $\tilde{\pi}_Y$.

Now $\tilde{\pi}_X, \tilde{\pi}_Y$ would be witnesses to (2), except that $\tilde{\pi}_Y$ is not necessarily continuous. However, $\tilde{\pi}_Y$ is continuous when restricted to an $E_0 \times I_{\mathbb{N}}$-invariant comeager Borel set C, so it suffices to find a continuous invariant embedding of $E_0 \times I_{\mathbb{N}}$ into $(E_0 \times I_{\mathbb{N}}) \restriction C$. One gets such an embedding by applying [Milc, Proposition 1.4] to the relation xRx' iff $x(E_0 \times I_{\mathbb{N}})x'$ or $x \notin C$ or $x' \notin C$. $\square$

Remark 4.2. *From the proof of (3) $\implies$ (2), one sees that if E is a countable Borel equivalence relation then actually one can strengthen (2) so that π_X is a continuous invariant embedding of $E_0 \times I_{\mathbb{N}}$ into E, i.e., a continuous embedding such that additionally $\pi_X([x]_{E_0 \times I_{\mathbb{N}}}) = [\pi_X(x)]_E$, for all $x \in 2^{\mathbb{N}} \times I_{\mathbb{N}}$.*

The next two results will be used in the proof of Theorem 1.8.

Theorem 4.3 (Theorem 1.9)**.** *Let F be a smooth Borel equivalence relation on a Polish space X, Y be a Polish space, and $P \subseteq X \times Y$ be a Borel set with countable sections. Suppose that*

$$\bigcap_{x \in C} P_x \neq \emptyset$$

for every F-class C. Then P admits a Borel F-invariant uniformization.

Proof. Let Z be a Polish space and $S : X \to Z$ be a Borel map such that $xFx' \iff S(x) = S(x')$. Define $P^* \subseteq Z \times Y$ by

$$P^*(z, y) \iff \forall x (S(x) = z \implies P(x, y)).$$

Note that P^* is $\mathbf{\Pi}^1_1$, and that if $S(x) = z$ then

$$P^*_z = \bigcap_{xFx'} P_{x'}$$

is non-empty and countable.

By Lusin-Novikov, fix a sequence g_n of Borel maps $g_n : X \to Y$ such that $P = \bigcup_n \operatorname{graph}(g_n)$. Define $Q(x, n) \iff P^*(S(x), g_n(x))$. Then Q is $\mathbf{\Pi^1_1}$, so by the number uniformization property [Kec95, 35.1] we can fix a Borel map h uniformizing Q.

Let now $A(z, y) \iff \exists x(S(x) = z \ \ \& \ \ y = g_{h(x)}(x))$. Then $A \subseteq P^*$ is $\mathbf{\Sigma^1_1}$, so by the Lusin separation theorem there is a Borel set $A \subseteq P^{**} \subseteq P^*$. By [Kec95, 18.9], the set

$$C = \{z \mid P^{**}_z \text{ is countable}\}$$

is $\mathbf{\Pi^1_1}$, and it contains $S(X)$, so by the Lusin separation theorem again there is some Borel set $S(X) \subseteq B \subseteq C$.

By Lusin-Novikov, there is a Borel uniformization f of $R(z, y) \iff B(z) \ \ \& \ \ P^{**}(z, y)$. Then $f \circ S$ is an F-invariant Borel uniformization of P. □

Proposition 4.4. *Let E be an analytic equivalence relation on a Polish space X, $F \supseteq E$ be a smooth Borel equivalence relation on X, Y be a Polish space, and $P \subseteq X \times Y$ be a Borel E-invariant set with countable sections. Suppose that*

$$xFx' \implies P_x \cap P_{x'} \neq \emptyset$$

for all $x, x' \in X$. Then there is a smooth equivalence relation $E \subseteq F' \subseteq F$ such that

$$\bigcap_{x \in C} P_x \neq \emptyset$$

for every F'-class C.

Proof. Let $G \subseteq X^{\mathbb{N}}$ be the $\aleph_0$-dimensional hypergraph of F-equivalent sequences x_n such that $\bigcap_n P_{x_n} = \emptyset$. By Lusin-Novikov, G is Borel.

We claim that G has a countable Borel colouring. By [Lec09, Lemma 2.1 and Theorem 1.6], it suffices to show that G has a countable $\sigma(\mathbf{\Sigma^1_1})$-colouring. Let S be a $\sigma(\mathbf{\Sigma^1_1})$-measurable selector for F and g_n be a sequence of Borel functions such that $P = \bigcup_n \operatorname{graph}(g_n)$. Then the function $f(x)$ assigning to x the least n such that $P(x, g_n(S(x)))$ is such a colouring. (In fact, $x \mapsto g_{f(x)}(S(x))$ is a $\sigma(\mathbf{\Sigma^1_1})$-measurable F-invariant uniformization of P.)

If A is G-independent, then so is $[A]_E$. Thus, by repeated application of the first reflection theorem, any G-independent analytic set is contained in an E-invariant G-independent Borel set. We may therefore fix a countable cover B_n of X by E-invariant G-independent Borel sets.

Define $xF'x' \iff xFx' \ \& \ \forall n(x \in B_n \iff x' \in B_n)$. Then F' is a smooth Borel equivalence relation and $E \subseteq F' \subseteq F$. Fix $x = x_0 \in X$, in order to show that

$$\bigcap_{xF'x'} P_{x'} \neq \emptyset.$$

Fix an enumeration $y_n, n \geq 1$ of P_x, and suppose for the sake of contradiction that this intersection is empty. Then for each n, there is some $x_n F' x$ with $y_n \notin P_{x_n}$. Also, $x \in B_k$ for some k. But then $x_n \in B_k$ for all k, so B_k is not G-independent, a contradiction. □

Proof of Theorem 1.8. Clearly the two cases are mutually exclusive. To see that at least one of them holds, define the graph G on X by $xGx' \iff P_x \cap P_{x'} = \emptyset$. By Lusin-Novikov, this is a Borel graph. We now apply the (G_0, H_0) dichotomy [Mil12, Theorem 25] to (G, E), and consider the two cases.

Case 1: There is a countable Borel colouring of $G \cap F$, where $F \supseteq E$ is smooth. Let A be Borel and $(G \cap F)$-independent. By repeated applications of the first reflection theorem, we may assume that A is E-invariant. We can therefore refine F to a smooth equivalence relation $F' \supseteq E$ such that $xF'x' \implies P_x \cap P_{x'} \neq \emptyset$. The result now follows from Theorem 4.3 and Proposition 4.4.

Case 2: Let f be a continuous homorphism from (G_0, H_0) to (G, E). It suffices to show that (4) holds in Proposition 4.1. To see this, consider $F = (f \times f)^{-1}(E), R = (f \times f)^{-1}(G)$. Then $H_0 \subseteq F$ and each F-section is G_0-independent, hence meager, so F is meager. We claim R is comeager. To see this, fix $x \in 2^{\mathbb{N}}$ and consider $R^c_x = \{x' : P_{f(x)} \cap P_{f(x')} \neq \emptyset\}$. Fix an enumeration y_n of $P_{f(x)}$, and let $A_n = \{x' : y_n \in P_{f(x')}\}$. Then each A_n is G_0-independent, hence meager, and $R^c_x = \bigcup_n A_n$. Thus R has comeager sections, and by Kuratowski-Ulam R is comeager. One can now recursively construct a continuous homomorphism g from $((\Delta_{2^{\mathbb{N}}})^c, E_0^c, E_0)$ to $((f \times f)^{-1}(\Delta_X)^c, R, E_0)$, see e.g. [Mila, Proposition 11]. Then $f \circ g$ satisfies (4). □

4.2 An $\aleph_0$-dimensional (G_0, H_0) dichotomy

In this section we state and prove an $\aleph_0$-dimensional analogue of Miller's (G_0, H_0) dichotomy [Mil12, Theorem 25]. This dichotomy generalizes Lecomte's $\aleph_0$-dimensional G_0 dichotomy [Lec09] (see also [Mil11]) in the same way that Miller's (G_0, H_0) dichotomy generalizes the G_0 dichotomy [KST99]. We then state an effective analogue of this theorem, and indicate the changes that must be made to prove it.

(A) Fix a strictly increasing sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ and dense sets $S \subseteq \bigcup_n \mathbb{N}^{2n}$, $T \subseteq \bigcup_n \mathbb{N}^{2n+1} \times \mathbb{N}^{2n+1}$, i.e., sets such that for all $u \in \mathbb{N}^{<\mathbb{N}}$ there is some $s \in S$ with $s \subseteq u$, and for all $(u, v) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}}$ there is some $t = (t_0, t_1) \in T$ such that $t_0 \subseteq u, t_1 \subseteq v$.

Let $X_\alpha = \{x \in \mathbb{N}^{\mathbb{N}} : \forall n \exists m \geq n(x{\restriction}m \in \alpha(m)^m)\}$. Note that X_α is dense G_δ in $\mathbb{N}^{\mathbb{N}}$.

Define the Borel $\aleph_0$-dimensional directed hypergraph G_0^ω on X_α by

$$G_0^\omega((x_n)) \iff \exists s \in S \exists z \in \mathbb{N}^{\mathbb{N}} \forall n(x_n = s^\frown n^\frown z),$$

and the Borel directed graph H_0^ω on X_α by

$$x H_0^\omega y \iff \exists (t_0, t_1) \in T \exists z \in \mathbb{N}^{\mathbb{N}}(x = t_0^\frown 0^\frown z \ \ \& \ \ y = t_1^\frown 1^\frown z).$$

We say $A \subseteq X_\alpha$ is G_0^ω-independent if $x \in A^{\mathbb{N}} \implies \neg G_0^\omega(x)$.

Proposition 4.5 ([Lec09, Lemma 2.1])**.** *Let $A \subseteq X_\alpha$ be Baire measurable and G_0^ω-independent. Then A is meager.*

Proof. Suppose A is non-meager, and fix an open set $N_s = \{x \in \mathbb{N}^{\mathbb{N}} : s \subseteq x\}$ in which A is comeager. By density of S, we may assume wlog that $s \in S$. For each n, the set $A_n = \{x \in \mathbb{N}^{\mathbb{N}} : s^\frown n^\frown x \in A\}$ is comeager, so there is some $x \in \bigcap_n A_n$. But then $x_n = s^\frown n^\frown x \in A$, and $G_0^\omega((x_n))$, so A is not G_0^ω-independent. $\square$

Let R be a quasi-order on a Polish space X. We let $\equiv_R$ denote the equivalence relation $x \equiv_R y \iff xRy \ \ \& \ \ yRx$. We say R is **lexicographically reducible** if there is a Borel reduction of R to the lexicographic order $\leq_{\text{lex}}$ on 2^α, for some $\alpha < \omega_1$. If $A \subseteq X$, we let $[A]^R = \{y : \exists x \in A(xRy)\}, [A]_R = \{y : \exists x \in A(yRx)\}$, and say A is closed upwards (resp. downward) for R if $A = [A]^R$ (resp. $A = [A]_R$). If $A, B \subseteq X$, we say (A, B) is R-independent if $A \times B \cap R = \emptyset$.

Proposition 4.6 (Ess. [Milb, Proposition 5])**.** *Let $A \subseteq X_\alpha$ be Baire measurable and $\equiv_{H_0^\omega}$-invariant. Then A is either meager or comeager.*

Proof. Suppose A is non-meager, and fix an open set N_u in which A is comeager. We show that A is non-meager in N_v for all $v \in \mathbb{N}^{<\mathbb{N}}$. By density of T, it suffices to show this assuming that $(u, v) \in T$. The set $A_0 = \{x \in \mathbb{N}^{\mathbb{N}} : u^\frown 0^\frown x \in A\}$ is comeager, and $x \in A_0 \implies v^\frown 1^\frown x \in A$, so A is comeager in $N_{v^\frown 1}$. $\square$

Proposition 4.7 ([Milb, Proposition 1]). *Let R be an analytic quasi-order on a Polish space X and $A_0, A_1 \subseteq X$ be analytic such that (A_0, A_1) is R-independent. Then there are Borel sets $A_i \subseteq B_i$ such that (B_0, B_1) is R-independent, B_0 is closed upwards for R and B_1 is closed downwards for R.*

Proof. Note that $([A_0]^R, [A_1]_R)$ is R-independent, and these sets are analytic. By the first reflection theorem, we can recursively construct a sequence of Borel sets B_n^i such that $A_i \subseteq B_0^i$, $[B_n^0]^R \subseteq B_{n+1}^0$, $[B_n^1]_R \subseteq B_{n+1}^1$, and (B_n^0, B_n^1) are R-independent. Take $B_i = \bigcup_n B_n^i$. $\square$

Let F be an equivalence relation on X and G be an $\aleph_0$-dimensional directed hypergraph on X. We call $A \subseteq X$ F-locally G-independent if there is no sequence $x_n \in A$ of pairwise F-equivalent points with $G((x_n))$, and we call $c : X \to Y$ an F-local colouring of G if $c^{-1}(y)$ is F-locally G-independent for all $y \in Y$.

Theorem 4.8. *Let G be an analytic $\aleph_0$-dimensional directed hypergraph on a Polish space X, and R an analytic quasi-order on X. Then exactly one of the following holds:*

(1) There is a lexicographically reducible quasi-order R' on X such that $R \subseteq R'$ and there is a countable Borel $\equiv_{R'}$-local colouring of G.

(2) There is a continuous homomorphism from (G_0^ω, H_0^ω) to (G, R).

Proof. To see these are mutually exclusive, it suffices to show that there is no smooth equivalence relation $F \supseteq \equiv_{H_0^\omega}$ such that there is a countable Borel F-local colouring $c : X_\alpha \to \mathbb{N}$ of G_0^ω. Arguing by contradiction, suppose such F, c existed. By Proposition 4.6, we can fix $n \in \mathbb{N}$ and a single F-class C such that $A = c^{-1}(n) \cap C$ is non-meager. But then by Proposition 4.5, A is not G_0^ω-independent, a contradiction.

We now show that at least one of these alternatives hold. Fix continuous maps $\pi_G, \pi_R : \mathbb{N}^\mathbb{N} \to X$ such that $G = \pi_G(\mathbb{N}^\mathbb{N}), R = \pi_R(\mathbb{N}^\mathbb{N})$. Let d denote the usual metric on $\mathbb{N}^\mathbb{N}$, and d_X be a complete metric compatible with the Polish topology on X.

Let V be a set, H_0 be an $\aleph_0$-dimensional directed hypergraph on V with edge set E_0, and H_1 be a directed graph on V with edge set E_1. A **copy** of (H_0, H_1) in (G, R) is a triple $\varphi = (\varphi_X, \varphi_G, \varphi_R)$ where $\varphi_X : V \to X, \varphi_G : E_0 \to \mathbb{N}^{\mathbb{N}}, \varphi_R : E_1 \to \mathbb{N}^{\mathbb{N}}$, such that

$$e = (v_n) \in E_0 \implies \varphi_G(e) = (\varphi_X(v_n))_{n\in\mathbb{N}},$$

and

$$e = (v, u) \in E_1 \implies \varphi_R(e) = (\varphi_X(v), \varphi_X(u)).$$

Let $\mathrm{Hom}(H_0, H_1; G, R)$ denote the set of all copies of (H_0, H_1) in (G, R). Note that if V, E_0, E_1 are countable, then $\mathrm{Hom}(H_0, H_1; G, R) \subseteq X^V \times (\mathbb{N}^{\mathbb{N}})^{E_0} \times (\mathbb{N}^{\mathbb{N}})^{E_1}$ is closed, hence Polish.

Suppose now we have H_0, H_1 as above, with V, E_0, E_1 countable, and consider $\mathcal{H} \subseteq \mathrm{Hom}(H_0, H_1; G, R)$. Let $\mathcal{H}(v) = \{\varphi_X(v) : \varphi \in \mathcal{H}\}$ for $v \in V$, and note that $\mathcal{H}(v)$ is analytic whenever $\mathcal{H}$ is analytic. Define $\mathcal{H}(e) \subseteq \mathbb{N}^{\mathbb{N}}$ similarly for $e \in E_0 \cup E_1$. Now call a set $\mathcal{H}$ **tiny** if it is Borel and there is a lexicographically reducible quasi-order R' on X such that $R \subseteq R'$ and one of the following holds:

(1) $\mathcal{H}(v)$ is $\equiv_{R'}$-locally G-independent for some $v \in V$.

(2) $\forall \varphi \in \mathcal{H} \exists u, v \in V(\varphi_X(u) \not\equiv_{R'} \varphi_X(v))$.

In this case, we call R' a **witness** that $\mathcal{H}$ is tiny, and say $\mathcal{H}$ is tiny of type 1 (resp. 2) if $\mathcal{H}$ satisfies (1) (resp. (2)). Finally, we say $\mathcal{H}$ is **small** if it is in the σ-ideal generated by the tiny sets, and otherwise we call $\mathcal{H}$ **large**.

Finally, fix H_0, H_1 as above with V, E_0, E_1 countable. For $v \in V$, we define the $\aleph_0$-dimensional directed hypergraph $\oplus_v H_0$ and the directed graph $\oplus_v H_1$ by taking a countable disjoint union of H_0 (resp. H_1), on vertex set $V \times \mathbb{N}$, and adding the edge $(v^\frown n)_{n\in\mathbb{N}}$ to $\oplus_v H_0$. Similarly, for $u, v \in V$, we define the $\aleph_0$-dimensional directed hypergraph $H_0\,{}_u{+}_v H_0$ and the directed graph $H_1\,{}_u{+}_v H_1$ by taking a countable disjoint union of H_0 (resp. H_1), on vertex set $V \times \mathbb{N}$, and adding the edge $(u^\frown 0, v^\frown 1)$ to $H_1\,{}_u{+}_v H_1$. Note that there are natural continuous projection maps

$$\mathrm{Hom}(\oplus_v H_0, \oplus_v H_1; G, R) \to \mathrm{Hom}(H_0, H_1; G, R)$$

and

$$\mathrm{Hom}(H_0\,{}_u{+}_v H_0, H_1\,{}_u{+}_v H_1; G, R) \to \mathrm{Hom}(H_0, H_1; G, R),$$

for all $n \in \mathbb{N}$, taking φ to its restriction φ^n to $V \times \{n\}$. If $\mathcal{H} \subseteq \mathrm{Hom}(H_0, H_1; G, R)$, we let

$$\oplus_v \mathcal{H} = \{\varphi \in \mathrm{Hom}(\oplus_v H_0, \oplus_v H_1; G, R) : \forall n(\varphi^n \in \mathcal{H})\},$$
$$\mathcal{H}\,{}_u{+}_v \mathcal{H} = \{\varphi \in \mathrm{Hom}(H_0\,{}_u{+}_v H_0, H_1\,{}_u{+}_v H_1; G, R) : \forall n(\varphi^n \in \mathcal{H})\}.$$

Claim 4.9. *If* $\mathrm{Hom}(\cdot,\cdot;G,R)$ *is small, then there is a lexicographically reducible quasi-order* R' *on* X *such that* $R \subseteq R'$ *and there is a countable Borel* $\equiv_{R'}$*-local colouring of* G*.*

Proof. Note that $\mathrm{Hom}(\cdot,\cdot;G,R)$ can be naturally identified with X, so that our assumption implies that X can be covered by countably-many Borel sets A_n such that for each n, there is a lexicographically reducible quasi-order R_n such that $R \subseteq R_n$ and A_n is $\equiv_{R_n}$-locally G-independent.

Let $f_n : X \to 2^{\alpha_n}$ be a Borel reduction of R_n to the lexicographic ordering on 2^{α_n}, $\alpha_n < \omega_1$. Let $\alpha = \sum_n \alpha_n$, and consider the map $f : X \to 2^\alpha$, $f(x) = f_0(x)^\frown f_1(x)^\frown f_2(n)^\frown \cdots$. Then f is Borel, so $xR'y \iff f(x) \leq_{\mathrm{lex}} f(y)$ is a lexicographically reducible quasi-order containing R. Note also that $\equiv_{R'} = \bigcap_n \equiv_{R_n}$. It follows that the map taking x to the least n for which $x \in A_n$ is a countable Borel $\equiv_{R'}$-local colouring of G. $\square$

Claim 4.10. *Let* H_0, H_1 *be as above with* V, E_0, E_1 *countable,* $F \subseteq V \cup E_0 \cup E_1$ *be finite,* $\varepsilon > 0$*, and* $\mathcal{H} \subseteq \mathrm{Hom}(H_0,H_1;G,R)$ *be large and Borel. Then there is a large Borel set* $\mathcal{H}' \subseteq \mathcal{H}$ *for which* $\mathrm{diam}_{d_X}(\mathcal{H}'(v)) < \varepsilon$ *for all* $v \in F \cap V$ *and* $\mathrm{diam}_d(\mathcal{H}'(e)) < \varepsilon$ *for all* $e \in F \cap (E_0 \cup E_1)$*.*

Proof. This follows from the fact that we can cover $X, \mathbb{N}^{\mathbb{N}}$ with countably many sets of small diameter, and the small sets form a σ-ideal. $\square$

Claim 4.11. *Let* H_0, H_1 *be as above with* V, E_0, E_1 *countable, and suppose* $\mathcal{H} \subseteq \mathrm{Hom}(H_0,H_1;G,R)$ *is Borel and large. Then* $\oplus_v \mathcal{H}, \mathcal{H}\,{}_u{+}_v \mathcal{H}$ *are Borel and large.*

Proof. That these sets are Borel is clear. Now suppose $\oplus_v \mathcal{H}$ is small and write $\oplus_v \mathcal{H} = \bigcup_{i \in 2, n \in \mathbb{N}} \mathcal{F}^i_n$, with F^i_n tiny of type i and witness R^i_n. Arguing as in the proof of Claim 4.9, we may assume that $R^i_n = R'$ for a single quasi-order R'. Let $v_n \in V$ be such that $\mathcal{F}^0_n(v_n)$ is $\equiv_{R'}$-locally G-independent. By the first reflection theorem, we may fix Borel sets $\mathcal{F}^0_n(v_n) \subseteq A_n$ which are $\equiv_{R'}$-locally G-independent. Define $\mathcal{H}_n = \{\varphi \in \mathcal{H} : \varphi_X(v_n) \in A_n\}$, and let

$$\mathcal{H}' = \mathcal{H} \setminus \left(\{\varphi \in \mathcal{H} : \exists u, v \in V (\varphi_X(u) \not\equiv_{R'} \varphi_X(v))\} \cup \bigcup_n \mathcal{H}_n \right).$$

We claim $\mathcal{H}'$ is tiny, which implies that $\mathcal{H}$ is small. Clearly $\mathcal{H}'$ is Borel, and we claim $\mathcal{H}'(v)$ is $\equiv_{R'}$-locally G-independent. Indeed, if $\varphi_n \in \mathcal{H}'$ and $G(((\varphi_n)_X(v))_{n\in\mathbb{N}})$, then there is some $\varphi \in \oplus_v \mathcal{H}$ with $\varphi^n = \varphi_n$ for all n. But then $\varphi \in \mathcal{F}^1_n$ for some n, so there are $u, w \in V \times \mathbb{N}$ such that $\varphi_X(u) \not\equiv_{R'} \varphi_X(w)$. Since $\varphi^n \in \mathcal{H}'$ for all n, we may assume that $u = v^\frown i, w = v^\frown j$ for some $i \neq j$. But then $\varphi^i_X(v) = (\varphi_i)_X(v) \not\equiv_{R'} (\varphi_j)_X(v) = \varphi^j_X(v)$.

Next suppose $\mathcal{H}\,{}_u{+}_v\mathcal{H}$ is small and write $\mathcal{H}\,{}_u{+}_v\mathcal{H} = \bigcup_{i\in 2, n\in\mathbb{N}} \mathcal{F}^i_n$, with $\mathcal{F}^i_n$ tiny of type i and witness R^i_n. As before, we may assume $R^i_n = R'$, and we define $\mathcal{H}_n, \mathcal{H}'$ in the same way, so that it suffices to show that $\mathcal{H}'$ is tiny of type 2.

Let $\varphi_i \in \mathcal{H}', i \in 2$, and suppose $(\varphi_0)_X(u) R (\varphi_1)_X(v)$. Then there is some $\varphi \in \mathcal{H}\,{}_u{+}_v\mathcal{H}$ with $\varphi^0 = \varphi_0$ and $\varphi^i = \varphi_1$ for $i > 0$. As before, we find that we must have $\varphi_X(u^\frown 0) \not\equiv_{R'} \varphi_X(v^\frown 1)$, so that $(\varphi_0)_X(u) \not\equiv_{R'} (\varphi_1)_X(v)$. Thus, $(\mathcal{H}'(u), \mathcal{H}'(v))$ is $(R\cap \equiv_{R'})$-independent, and by Proposition 4.7 we can find Borel sets $\mathcal{H}'(u) \subseteq A, \mathcal{H}'(v) \subseteq B$ such that A is closed upwards for $R\cap \equiv_{R'}$, B is closed downwards for $R\cap \equiv_{R'}$, and (A, B) is $(R\cap \equiv_{R'})$-independent. Then

$$xQy \iff xR'y \ \&\ (x \equiv_{R'} y \ \&\ x \in A \implies y \in A)$$

is a lexicographically reducible quasi-order containing R, and $\mathcal{H}'$ is tiny of type 2 with witness Q. $\square$

If $\mathrm{Hom}(\cdot,\cdot; G, R)$ is small, then by Claim 4.9 we are done. Suppose now that $\mathrm{Hom}(\cdot,\cdot; G, R)$ is large. We define a sequence G_n of $\aleph_0$-dimensional directed graphs on $\mathbb{N}^n$ and a sequence H_n of directed graphs on $\mathbb{N}^n$ as follows:

$$\begin{aligned} G_n(x_i) &\iff \exists k < n\, \exists s \in (S \cap \mathbb{N}^k)\, \exists u \in \mathbb{N}^{n-k-1}\, \forall i (x_i = s^\frown i^\frown u), \\ xH_ny &\iff \exists k < n\, \exists (t_0, t_1) \in (T \cap \mathbb{N}^k \times \mathbb{N}^k) \\ &\qquad\qquad \exists u \in \mathbb{N}^{n-k-1} (x = t_0^\frown 0^\frown u \ \&\ y = t_1^\frown 1^\frown y). \end{aligned}$$

Note that if $s \in S \cap \mathbb{N}^n$ then $G_{n+1} = \oplus_s G_n$ and $H_{n+1} = \oplus_s H_n$, and if $(t_0, t_1) \in T \cap \mathbb{N}^n \times \mathbb{N}^n$ then $G_{n+1} = G_n\,{}_{t_0}{+}_{t_1} G_n$ and $H_{n+1} = H_n\,{}_{t_0}{+}_{t_1} H_n$. Also,

$$G^\omega_0((x_i)_{i\in\mathbb{N}}) \iff \exists N \forall n \geq N (G_n((x_i {\restriction} n)_{i\in\mathbb{N}}))$$

and

$$xH^\omega_0 y \iff \exists N \forall n \geq N (x{\restriction}n\, H_n\, y{\restriction}n),$$

and G_n, H_n have countably many vertices and edges.

By Claims 4.10 and 4.11, we can recursively construct a sequence of large Borel sets $\mathcal{H}_n \subseteq \mathrm{Hom}(G_n, H_n; G, R)$ such that $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n \oplus_s \mathcal{H}_n$ for $s \in S\cap\mathbb{N}^n$, $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n\,{}_{t_0}{+}_{t_1}\mathcal{H}_n$ for $(t_0, t_1) \in T \cap \mathbb{N}^n \times \mathbb{N}^n$, $\mathrm{diam}_{d_X}(\mathcal{H}_n(x)) < 2^{-n}$ for all $x \in \alpha(n)^n$, and $\mathrm{diam}_d(\mathcal{H}(e)) < 2^{-n}$ for all $e \in G_n \cup H_n$ with $e_0 \in \alpha(n)^n$, where e_0 denotes the first vertex in e. It follows that $\{f(x)\} = \bigcap_n \overline{\mathcal{H}_n(x \restriction n)}$ exists and is well defined for $x \in X_\alpha$, and that this map $f : X_\alpha \to X$ is continuous. To see that it is a homomorphism of G_0^ω to G, suppose $G_0^\omega((x_i)_{i\in\mathbb{N}})$ and let N be sufficiently large that $G_N((x_i \restriction N)_{i\in\mathbb{N}})$. Then $\{y\} = \bigcap_{n\geq N} \overline{\mathcal{H}_n((x_i \restriction n)_{i\in\mathbb{N}})}$ exists and is well defined, and by continuity we have $(f(x_i))_{i\in\mathbb{N}} = \pi_G(y) \in G$. A similar argument shows that f is a homomorphism from H_0^ω to R. $\square$

(B) This dichotomy admits the following effective refinement:

Theorem 4.12. *Let G be a Σ^1_1 $\aleph_0$-dimensional directed hypergraph on a Polish space X, and R a Σ^1_1 partial order on X. Then exactly one of the following holds:*

1. *There is a quasi-order R' on X such that $R \subseteq R'$, there is a countable Δ^1_1 $\equiv_{R'}$-local colouring of G, and there is a Δ^1_1 reduction of R' to the lexicographic order $\leq_{lex}$ on 2^α, for some $\alpha < \omega_1^{CK}$.*

2. *There is a continuous homomorphism from (G_0^ω, H_0^ω) to (G, R).*

To prove this, we make the following modifications to the proof of Theorem 4.8. First, we choose π_G, π_H to be computable (restricting their domains appropriately to Π^0_1 sets). We then replace “Borel” with “Δ^1_1” and “lexicographically reducible” with “admitting a Δ^1_1 reduction to $\leq_{\text{lex}}$ on 2^α, for some $\alpha < \omega_1^{CK}$” in the definition of tiny sets.

We now have the following:

Lemma 4.13. *Let V be a set, H_0 be an $\aleph_0$-dimensional directed hypergraph on V with edge set E_0 and H_1 be a directed graph on V with edge set E_1, with V, E_0, E_1 countable. Suppose $\mathcal{H} \subseteq \mathrm{Hom}(H_0, H_1; G, R)$ is small and Δ^1_1. Then one can find:*

(1) a uniformly Δ^1_1 sequence of tiny sets $(\mathcal{F}^i_n)_{i\in 2, n\in\mathbb{N}}$ covering $\mathcal{H}$,
(2) a uniformly Δ^1_1 sequence $(R^i_n)_{i\in 2, n\in\mathbb{N}}$ of quasi-orders on $\mathbb{N}$,
(3) a uniformly Δ^1_1 sequence of ordinals $\alpha^i_n < \omega_1^{CK}$,
(4) a uniformly Δ^1_1 sequence $(f^i_n)_{i\in 2, n\in\mathbb{N}}$ of maps $f^i_n : \mathbb{N}^\mathbb{N} \to 2^{\alpha^i_n}$, and
(5) a uniformly Δ^1_1 sequence $v_n \in V$,

such that the sets $\mathcal{F}^i_n$ are pairwise disjoint, $R \subseteq R^i_n$ for all i, n, each f^i_n is a reduction of R^i_n to $\leq_{lex}$ on $2^{\alpha^i_n}$, $\mathcal{F}^0_n(v_n)$ are $\equiv_{R^0_n}$-locally G-independent, and $\forall \varphi \in \mathcal{H} \exists u, v \in V(\varphi_X(u) \not\equiv_{R^1_n} \varphi_X(v))$.

Proof sketch. Fix a nice coding $D \ni n \mapsto D_n$ of the Δ^1_1 sets. The assertion that $(\mathcal{F}, R', \alpha, f, v, i)$ is a witness that $\mathcal{F}$ is tiny of type i is Π^1_1-on-Δ^1_1. It follows that the relation "$\varphi \notin \mathcal{H}$ or $n \in D$ codes such a tuple with $\varphi \in \mathcal{F}$" is Π^1_1, and hence by the number uniformization theorem for Π^1_1 there is a Δ^1_1 map $g : \mathcal{H} \to D$ taking each $\varphi \in \mathcal{H}$ to such a tuple. The image of $\mathcal{H}$ under g is Σ^1_1, so by the Lusin separation theorem there is a Δ^1_1 set $A \subseteq D$ containing $g(\mathcal{H})$ and such that every element in A codes a tuple as above. One can then fix a Δ^1_1 enumeration of A, which satisfies all of the above conditions except maybe pairwise disjointness of the family $\mathcal{F}^i_n$, and this can be fixed by a straightforward recursive construction. □

The effective analogue of Claim 4.9 follows immediately. We note that the first reflection theorem is effective enough that the proof of Proposition 4.7 is effective as well. Claim 4.11 now follows using Lemma 4.13. The rest of the proof is identical to that of Theorem 4.8.

4.3 Proof of Theorem 1.8 from the $\aleph_0$-dimensional (G_0, H_0) dichotomy

(A) Clearly the two cases are mutually exclusive. To see that at least one of them holds, define the $\aleph_0$-dimensional hypergraph G on X by $G(x_n) \iff \bigcap_n P_{x_n} = \emptyset$. By Lusin-Novikov, G is Borel. We now apply Theorem 4.8 to (G, E), and consider the two cases.

Case 1: There is a lexicographically reducible quasi-order R containing E and a countable Borel $\equiv_R$-local colouring of G. Let $F = \equiv_R$, so that $E \subseteq F$ and F is smooth. Since P is E-invariant, if A is F-locally G-independent then so is $[A]_E$. It follows that there is a countable Borel E-invariant F-local colouring of G, so that after refining F with this colouring we may assume that X is F-locally G-independent, i.e., $\bigcap_{x \in C} P_x \neq \emptyset$ for every F-class C. Then P admits a Borel F-invariant uniformization by Theorem 4.3.

Case 2: There is a continuous homomorphism $\pi : X_\alpha \to X$ of (G_0^ω, H_0^ω) to (G, E). We will show that (4) holds in Proposition 4.1. To see this, consider $F = (\pi \times \pi)^{-1}(E)$ and $R = (\pi \times \pi)^{-1}(R')$, where $x R' x' \iff P_x \cap P_{x'} = \emptyset$. Note that R' is Borel by Lusin-Novikov, and hence so is R. Also, $H_0^\omega \subseteq F$ and $F \cap R = \emptyset$.

We claim that R is comeager. To see this, fix $x \in X_\alpha$ and consider

$$R^c_x = \{x' \in X_\alpha : P_{\pi(x)} \cap P_{\pi(x')} \neq \emptyset\}.$$

Fix an enumeration y_n of $P_{\pi(x)}$, and let $A_n = \{x' \in X_\alpha : y_n \in P_{\pi(x')}\}$. Then each A_n is G_0^ω-independent, hence meager, and hence so is $R_x^c = \bigcup_n A_n$. Thus R_x is comeager for all $x \in X_\alpha$, and by Kuratowski-Ulam R is comeager.

One can now recursively construct a continuous homomorphism $f : 2^\omega \to X_\alpha$ from $(\Delta(2^\omega)^c, E_0^c, E_0)$ to $((\pi \times \pi)^{-1}(\Delta(X))^c, R, F)$, see e.g. [Mila, Proposition 11]. Then $\pi \circ f$ satisfies (4).

(B) We note the following effective version of Theorem 1.8:

Theorem 4.14. *Let E be a Δ_1^1 equivalence relation on $\mathbb{N}^\mathbb{N}$ and $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ an E-invariant Δ_1^1 relation with countable non-empty sections. Then exactly one of the following holds:*

1. *There is a Δ_1^1 E-invariant uniformization,*

2. *There is a continuous embedding $\pi_X \colon 2^\mathbb{N} \times \mathbb{N} \to X$ of $E_0 \times I_\mathbb{N}$ into E and a continuous injection $\pi_Y \colon 2^\mathbb{N} \times \mathbb{N} \to Y$ such that for all $x, x' \in 2^\mathbb{N} \times \mathbb{N}$,*

$$\neg(x \; E_0 \times I_\mathbb{N} \; x') \implies P_{\pi_X(x)} \cap P_{\pi_X(x')} = \emptyset$$

and

$$P_{\pi_X(x)} = \pi_Y([x]_{E_0 \times I_\mathbb{N}}).$$

This follows from the above proof, Theorem 4.12, and the fact that the proof of Theorem 4.3 is effective.

4.4 Proof of Theorem 1.10

(A) Note first that (1) is equivalent to the existence of a smooth Borel equivalence $F \supseteq E$ for which P is F-invariant, by Theorem 2.2.

To see that these are mutually exclusive, let $F \supseteq E$ be smooth so that P is F-invariant, and suppose that π_X, π_Y witness (2). Then there is a comeagre E_0-invariant set C that π_X maps into a single F-class, so $\pi_Y(C)$ is contained in a single P-section, a contradiction.

Now define the graph $xGx' \iff P_x \neq P_{x'}$. This graph is Borel by Lusin-Novikov. Apply the (G_0, H_0) dichotomy to (G, E).

Case 1: There is a smooth $F \supseteq E$ such that G admits a countable Borel F-local colouring. If A is analytic and F-locally G-independent, then so is $[A]_E$, so by repeated applications of the first reflection theorem it is contained in a Borel E-invariant F-locally G-independent set. Thus we may assume that G admits a countable Borel E-invariant F-local colouring, and hence by refining F with this colouring we may assume that actually $G \cap F = \emptyset$, i.e., P is F-invariant. Thus (1) holds.

Case 2: There is a continuous homomorphism $\varphi : 2^{\mathbb{N}} \to X$ of (G_0, H_0) to (G, E). Define $R(x, y) \iff P(\varphi(x), y)$, and let

$$Q(x, y) \iff R(x, y) \ \& \ \forall^* x' \neg R(x', y),$$

where $\forall^* x A(x)$ means A is comeager for $A \subseteq 2^{\mathbb{N}}$. Let $A = \operatorname{proj}(Q)$ and $xSx' \iff Q_x \cap Q_{x'} \neq \emptyset$. Then R is Borel with countable sections, and it follows that Q, A, S are Borel as well. Additionally, R, Q are E_0-invariant.

We claim that A is comeager and S is meager. Granted this, we can find a continuous homomorphism $\psi : 2^{\mathbb{N}} \to A$ of (E_0, E_0^c) to (E_0', S^c) such that $\varphi \circ \psi$ is injective, where E_0' is the smallest equivalence relation containing H_0 (cf. [Mila, Proposition 11]). Now the set $Q'(x, y) \iff Q(\psi(x), y)$ has countable sections, so it admits a Borel uniformization g. Since ψ is a homomorphism from E_0^c to S^c, g is countable-to-one, so by Lusin-Novikov it is injective and continuous on a non-meager set B. Let τ be a continuous embedding of E_0 into $E_0 \restriction B$. Then $\pi_X = \varphi \circ \psi \circ \tau, \pi_Y = g \circ \tau$ satisfy (2).

Now suppose that A is comeager, in order to show that S is meager. By Kuratowski-Ulam, it suffices to show that S_x is meager for all $x \in A$. So consider $x \in A$, and let $y \in Q_x$ be arbitrary. Then $y \notin R_{x'}$ for comeagerly-many x', and so $y \notin Q_{x'}$ for comeagerly-many x'. Since Q_x is countable, it follows that $Q_x \cap Q_{x'} = \emptyset$ for comeagerly-many x', and so S_x is meager.

It remains to show that A is comeager. To see this, define $xBx' \iff R_x \subseteq R_{x'}$. For any x, $B_x = \bigcap_{y \in R_x} \{x' : R(x', y)\}$, so if B_x is meager then there is some $y \in R_x$ for which $\{x' : R(x', y)\}$ is not comeager. But this set is Borel and E_0-invariant, so it is meager, and hence $y \in Q_x$ and $x \in A$. Thus by Kuratowski-Ulam it suffices to show that B is meager.

Suppose for the sake of contradiction that B is non-meager. Let C be the set of all x so that B_x is non-meager. Since B_x is E_0-invariant, it must be comeager for all $x \in C$. Moreover C is non-meager and E_0-invariant, hence it is comeager. It follows that B is comeager, and hence so is $B'(x, x') \iff B(x, x') \ \& \ B(x', x)$. In particular, $B'_x = C$ is comeager for some x. But then $x, x' \in C \implies R_x = R_{x'}$, hence C is G_0-independent, a contradiction.

Remark 4.15. *This proof actually shows that in case (2), we can take π_X, π_Y so that additionally $\pi_Y(x) \in P_{\pi_X(x')} \iff xE_0x'$.*

(B) This proof can also be made effective, by Theorem 4.12:

Theorem 4.16. *Let E be a Δ^1_1 equivalence relation on X and $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ an E-invariant Δ^1_1 relation with countable non-empty sections. Then exactly one of the following holds:*

(1) There is a uniformly Δ^1_1 sequence $g_n : X \to Y$ of E-invariant uniformizations with $P = \bigcup_n \operatorname{graph}(g_n)$,

(2) There is a continuous embedding $\pi_X : 2^{\mathbb{N}} \to X$ of E_0 into E and a continuous injection $\pi_Y : 2^{\mathbb{N}} \to Y$ such that for all $x \in 2^{\mathbb{N}}$, $P(\pi_X(x), \pi_Y(x))$.

4.5 Proofs of Proposition 1.11 and Theorem 1.12

Let us fix a parametrization of the Borel relations on $\mathbb{N}^{\mathbb{N}}$, as in [AK00, Section 5] (see also [Mos09, Section 3H]). This consists of a set $D \subseteq 2^{\mathbb{N}}$ and two sets $S, P \subseteq (\mathbb{N}^{\mathbb{N}})^3$ such that

(i) D is Π^1_1, S is Σ^1_1 and P is Π^1_1;

(ii) for $d \in D$, $S_d = P_d$, and we denote this set by D_d;

(iii) every Borel set in $(\mathbb{N}^{\mathbb{N}})^2$ appears as D_d for some $d \in D$; and

(iv) if $B \subseteq X \times (\mathbb{N}^{\mathbb{N}})^2$ is Borel, X a Polish space, there is a Borel function $p : X \to 2^{\mathbb{N}}$ so that $B_x = D_{p(x)}$ for all $x \in X$.

Define

$$\mathcal{P} = \{(d, e) : D_d \text{ is an equivalence relation on } \mathbb{N}^{\mathbb{N}} \text{ and } D_e \text{ is } D_d\text{-invariant}\},$$

and let $\mathcal{P}^{unif}$ denote the set of pairs $(d, e) \in \mathcal{P}$ for which D_e admits a D_d-invariant uniformization. More generally, for any set A of properties of sets $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, let $\mathcal{P}_A$ (resp. $\mathcal{P}^{unif}_A$) denote the set of pairs (d, e) in $\mathcal{P}$ (resp. $\mathcal{P}^{unif}$) such that D_e satisfies all of the properties in A. Let $\mathcal{P}_{ctble}$ (resp. $\mathcal{P}^{unif}_{ctble}$) denote $\mathcal{P}_A$ (resp. $\mathcal{P}^{unif}_A$) for A consisting of the property that P has countable sections.

We are interested in properties asserting that D_e, or its sections, are G_δ, F_σ, comeager, non-meager, μ-positive, μ-conull, countable, or K_σ, where μ varies over probability Borel measures on $\mathbb{N}^{\mathbb{N}}$. It is straightforward to check, using [Kec95, 16.1, 17.25, 18.9, 35.47], that for all such sets of properties A, $\mathcal{P}_A$ is $\mathbf{\Pi^1_1}$ and $\mathcal{P}^{unif}_A$ is $\mathbf{\Sigma^1_2}$.

By Theorem 4.14, we can bound the complexity of $\mathcal{P}^{unif}_{ctble}$:

Proposition 4.17 (Proposition 1.11)**.** *The set $\mathcal{P}^{unif}_{ctble}$ is $\mathbf{\Pi^1_1}$.*

Proof. By Theorem 4.14, $(d,e) \in \mathcal{P}^{unif}_{ctble}$ iff $(d,e) \in \mathcal{P}_{ctble}$ and there exists a $\Delta^1_1(d,e)$ function f which is a D_d-invariant uniformization of D_e. The assertion that a $\Delta^1_1(d,e)$ function f is a D_d-invariant uniformization of D_e is $\Pi^1_1(d,e)$, so by bounded quantification for Δ^1_1 [Mos09, 4D.3], $\mathcal{P}^{unif}_{ctble}$ is Π^1_1. $\square$

Recall that a set B in a Polish space X is called $\mathbf{\Sigma^1_2}$-complete if it is $\mathbf{\Sigma^1_2}$, and for all zero-dimensional Polish spaces Y and $\mathbf{\Sigma^1_2}$ sets $C \subseteq Y$ there is a continuous function $f : Y \to X$ such that $C = f^{-1}(B)$. Note that by [Paw14], one could equivalently take f to be Borel in this definition.

The following computes the exact complexity of the sets $\mathcal{P}^{unif}_A$, when A asserts that D_e has "large" sections.

Theorem 4.18 (Theorem 1.12)**.** *The set $\mathcal{P}^{unif}_A$ is $\mathbf{\Sigma^1_2}$-complete, where A is one of the following sets of properties of $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$:*

1. *P has non-meager sections;*

2. *P has non-meager G_δ sections;*

3. *P has non-meager sections and is G_δ;*

4. *P has μ-positive sections for some probability Borel measure μ on $\mathbb{N}^{\mathbb{N}}$;*

5. *P has μ-positive F_σ sections for some probability Borel measure μ on $\mathbb{N}^{\mathbb{N}}$;*

6. *P has μ-positive sections for some probability Borel measure μ on $\mathbb{N}^{\mathbb{N}}$ and is F_σ.*

The same holds for comeager instead of non-meager, and μ-conull instead of μ-positive.

In fact, there is a hyperfinite Borel equivalence relation E with code $d \in D$ such that for all such A above, the set of $e \in D$ such that $(d,e) \in \mathcal{P}^{unif}_A$ is $\mathbf{\Sigma^1_2}$-complete.

Proof. We will show this first when A asserts that P is G_δ and has comeager sections. Since $\mathbb{N}^{\mathbb{N}}$ is Borel isomorphic to $\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$, we may assume that D_d is instead an equivalence relation on $\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$, and that $D_e \subseteq (\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}}$.

Let E be the hyperfinite Borel equivalence relation on $\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ given by

$$(x, y) E (x', y') \iff x = x' \ \& \ y E_0 y',$$

fix a code $d \in D$ for E, and let $\mathcal{P}_A^{unif}(E)$ denote the set of all $e \in D$ so that $(d, e) \in \mathcal{P}_A^{unif}$. We will show that $\mathcal{P}_A^{unif}(E)$ is $\mathbf{\Sigma}^1_2$-complete.

Let now T be a tree on $\mathbb{N} \times \mathbb{N}$ (cf. [Kec95, 2.C]). Each such tree T defines a closed subset $[T] \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ given by

$$[T] = \{(x, y) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : \forall n \, ((x \restriction n, y \restriction n) \in T)\}.$$

We say $[T]$ admits a **full Borel uniformization** if there is a Borel map $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ so that $(x, f(x)) \in [T]$ for all $x \in \mathbb{N}^{\mathbb{N}}$, and we denote by FBU the set of trees on $\mathbb{N} \times \mathbb{N}$ which admit full Borel uniformizations.

By the proof of Theorem 1.5, and considering $\mathbb{N}^{\mathbb{N}}$ as a co-countable set in $2^{\mathbb{N}}$, there is a G_δ set $P \subseteq 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with comeager sections which is E_0 invariant, and so that

$$\bigcap_{x \in C} P_x = \emptyset$$

whenever $C \subseteq 2^{\mathbb{N}}$ is μ-positive, where μ is the uniform product measure on $2^{\mathbb{N}}$. Given a tree T on $\mathbb{N} \times \mathbb{N}$, define $P_T \subseteq (\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}}$ by

$$P_T(x, y, z) \iff P(y, z) \vee (x, z) \in [T].$$

Note that P_T is G_δ, E-invariant, and has comeager sections.

Claim 4.19. *$[T]$ admits a full Borel uniformization iff P_T admits a Borel E-invariant uniformization.*

Proof. If f is a full Borel uniformization of $[T]$, then $g(x, y) = f(x)$ is an E-invariant Borel uniformization of P_T. Conversely, suppose g were an E-invariant Borel uniformization of P_T. For $x \in \mathbb{N}^{\mathbb{N}}$, let $g_x(y) = g(x, y)$. Then $g_x : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is E_0-invariant, hence constant on a μ-conull set $C \subseteq 2^{\mathbb{N}}$. Since

$$\bigcap_{y \in C} P_y = \emptyset,$$

we cannot have $P(y, g_x(y))$ for all $y \in C$, and so $(x, g_x(y)) \in [T]$ for all $y \in C$. Thus

$$f(x) = z \iff \forall^*_\mu y (g(x, y) = z)$$

is a full Borel uniformization of $[T]$ (cf. [Kec95, 17.26] and the paragraphs following it). $\square$

By identifying trees on $\mathbb{N} \times \mathbb{N}$ with their characteristic functions, we can view the space of trees as a closed subset of $2^{\mathbb{N}}$. The set B given by

$$B(T, x, y, z) \iff T \text{ is a tree and } P_T(x, y, z)$$

is clearly Borel, so there is a Borel map p such that for each tree T, $p(T) \in D$ and $D_{p(T)} = P_T$. It follows by Claim 4.19 that FBU $= p^{-1}(\mathcal{P}_A^{unif}(E))$. By [AK00, Lemma 5.3], the set FBU is $\mathbf{\Sigma}_2^1$-complete, and hence so is $\mathcal{P}_A^{unif}(E)$.

The cases 1–3 follow from this as well. For 4–6, simply replace P in the above proof with an F_σ set $Q \subseteq 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with μ-conull sections which is E_0-invariant, and so that

$$\bigcap_{x \in C} Q_x = \emptyset$$

whenever $C \subseteq 2^{\mathbb{N}}$ is non-meager, which exists by the proof of Theorem 1.5. $\square$

Remark 4.20. *We do not know the complexity of $\mathcal{P}_A^{unif}$ when A asserts that P is G_δ and has comeager μ-conull sections for a probability Borel measure μ. By the proof of Theorem 1.6, there is an E_0-invariant G_δ set $R \subseteq [\mathbb{N}]^{\aleph_0} \times \mathbb{N}^{\mathbb{N}}$ with comeager μ-conull sections, such that*

$$\bigcap_{x \in C} P_x = \emptyset$$

for all Ramsey-positive sets $C \subseteq [\mathbb{N}]^{\aleph_0}$. One can define P_T for a tree T on $\mathbb{N} \times \mathbb{N}$ as in the proof of Theorem 1.12, however the "if" direction of our proof of Claim 4.19 no longer works (cf. [Sab12]).

4.6 Proof of Proposition 1.14

By [Kec95, 18.17], there is a G_δ set $R \subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $\mathrm{proj}_{\mathbb{N}^{\mathbb{N}}}(R) = \mathbb{N}^{\mathbb{N}}$ which does not admit a Borel uniformization. Write $R = \bigcap_n Q_n$, $Q_n \subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ open, and define P by

$$P(n, x, y) \iff Q_n(x, y).$$

Let $(n, x) F (m, x') \iff x = x'$. Then F is a smooth countable Borel equivalence relation, P is open, and if $C = [(n, x)]_F$ is an F-class then

$$\bigcap_{u \in C} P_u = \bigcap_n P_{(n,x)} = \bigcap_n (Q_n)_x = R_x \neq \emptyset.$$

Suppose now towards a contradiction that $g : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ is an F-invariant uniformization of P. Define $f : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $f(x) = g(0, x)$. Then $f(x) = g(0, x) = g(n, x) \in P_{(n,x)}$ for all n, so $f(x) \in \bigcap_n P_{(n,x)} = R_x$, a contradiction.

5 On Conjecture 1.15

Concerning Conjecture 1.15, we first note the following analog of Lemma 2.1.

Lemma 5.1. *Let E, F be Borel equivalence relations on Polish spaces X, X', resp., such that $E \leq_B E'$. If E fails (b) (resp., (c), (d)), so does E'.*

The proof is identical to that of Lemma 2.1. Note now that any countable Borel equivalence relation E trivially satisfies (b), (c), and (d), so by Lemma 5.1, in Conjecture 1.15, (a) implies (b), (c) and (d).

To verify then Conjecture 1.15, one needs to show that if E is not reducible to countable, then (b), (c) and (d) fail. It is an open problem (see [HK01, end of Section 6]) whether the following holds:

Problem 5.2. *Let E be a Borel equivalence relation which is not reducible to countable. Then one of the following holds:*

(1) $E_1 \leq_B E$, where E_1 is the following equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$:

$$xE_1y \iff \exists m \forall n \geq m (x_n = y_n);$$

(2) There is a Borel equivalence relation F induced by a turbulent continuous action of a Polish group on a Polish space such that $F \leq_B E$;

(3) $E \leq_B E$, where $E_0^{\mathbb{N}}$ is the following equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$:

$$xE_0^{\mathbb{N}}y \iff \forall n (x_n E_0 y_n).$$

It is therefore interesting to show that (b), (c) and (d) fail for E_1, F as in (2) above, and $E_0^{\mathbb{N}}$. Here are some partial results.

Proposition 5.3. *Let E be a Borel equivalence relation which is not reducible to countable but is Borel reducible to a Borel equivalence relation F with K_σ classes. Then E fails (d). In particular, E_1 and E_2 fail (d), where E_2 is the following equivalence relation on $2^{\mathbb{N}}$:*

$$xE_2y \iff \sum_{n : x_n \neq y_n} \frac{1}{n+1} < \infty.$$

Proof. Suppose E, F live on the Polish spaces X, Y, resp., and let $g\colon X \to Y$ be a Borel reduction of E to F. Define $P \subseteq X \times X$ as follows:

$$(x, y) \in P \iff g(x) F y.$$

Clearly P is E-invariant and has K_σ sections. Suppose then that P admitted a Borel E-invariant countable uniformization $f\colon X \to Y^{\mathbb{N}}$. Then define $h\colon X \to X$ by $g(x) = f(x)_0$. Then by [Kec24, Proposition 3.7], h shows that E is reducible to countable, a contradiction. $\square$

Concerning (b) and (c) for E_1, the following is a possible example for their failure.

Problem 5.4. *Let $X = (2^{\mathbb{N}})^{\mathbb{N}}, Y = 2^{\mathbb{N}}$ and define $P \subseteq X \times Y$ as follows:*

$$(x, y) \in P \iff \exists m \forall n \geq m (x_n \neq y),$$

so that P is E_1-invariant and each section P_x is co-countable, so has μ-measure 1 (for μ the product measure on Y) and is comeager. Is there a Borel E_1-invariant countable uniformization of P?

One can show the following weaker result, which provides a Borel anti-diagonalization theorem for E_1.

Proposition 5.5. *Let $f\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel function such that $x E_1 y \implies f(x) = f(y)$. Then there is $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that for infinitely many n, $f(x) = x_n$.*

Thus if X, Y, P are as in Problem 5.4, P does not admit a Borel E_1-invariant uniformization.

Proof. For any nonempty countable set $S \subseteq 2^{\mathbb{N}}$ consider the product space $S^{\mathbb{N}}$ with the product topology, where S is taken to be discrete. Denote by $E_0(S)$ the equivalence relation on $S^{\mathbb{N}}$ given by $x E_0(S) y \iff \exists m \forall n \geq m (x_n = y_n)$. This is generically ergodic and for $x, y \in S^{\mathbb{N}}$ we have that $x E_0(S) y \implies f(x) = f(y)$, so there is (unique) $x_S \in 2^{\mathbb{N}}$ such that $f(x) = x_S$, for comeager many $x \in S^{\mathbb{N}}$. Clearly x_S can be computed in a Borel way given any $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ with $S = \{x_n \colon n \in \mathbb{N}\}$, i.e., we have a Borel function $F\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that

$$\{x_n \colon n \in \mathbb{N}\} = \{y_n \colon n \in \mathbb{N}\} = S \implies F((x_n)) = F((y_n)) = x_S.$$

We now use the following Borel anti-diagonalization theorem of H. Friedman, see [Sta85, Theorem 2, page 23]:

Theorem 5.6 (H. Friedman)**.** *Let E be a Borel (even analytic) equivalence relation on a Polish space X. Let $F\colon X^{\mathbb{N}} \to X$ be a Borel function such that*

$$\{[x_n]_E \colon n \in \mathbb{N}\} = \{[y_n]_E \colon n \in \mathbb{N}\} \implies F((x_n)) \ E \ F((y_n)).$$

Then there is $x \in X^{\mathbb{N}}$ and $i \in \mathbb{N}$ such that $F(x) E x_i$.

Applying this to E being the equality relation on $2^{\mathbb{N}}$ and F as above, we conclude that for some S, we have that $x_S \in S$. Then for comeager many $x \in S^{\mathbb{N}}$ we have that $x_n = x_S$, for infinitely many n, and also $(x, x_S) \in P$, a contradiction. □

In response to a question by Andrew Marks, we note the following version of Proposition 5.5 for E_1 restricted to injective sequences. Below $[2^{\mathbb{N}}]^{\mathbb{N}}$ is the Borel subset of $(2^{\mathbb{N}})^{\mathbb{N}}$ consisting of injective sequences and $x \leq_T y$ means that x is recursive in y.

Proposition 5.7. *Let $g\colon [2^{\mathbb{N}}]^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel function such that $xE_1y \implies g(x) = g(y)$. Then there is $y \in [2^{\mathbb{N}}]^{\mathbb{N}}$ such that for all n, $g(y) \leq_T y_n$.*

Proof. Fix a recursive bijection $x \mapsto \langle x \rangle$ from $(2^{\mathbb{N}})^{\mathbb{N}}$ to $2^{\mathbb{N}}$ and for each $i \in \mathbb{N}$ let $\bar{i} \in 2^{\mathbb{N}}$ be the characteristic function of $\{i\}$. Then for each $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $i \in \mathbb{N}$, put

$$\bar{x}^i = \langle \bar{i}, x_i, x_{i+1}, \dots \rangle \in 2^{\mathbb{N}}.$$

and

$$x' = \langle \bar{x}^0, \bar{x}^1, \dots \rangle \in [2^{\mathbb{N}}]^{\mathbb{N}}.$$

Note that $xE_1y \implies x'E_1y'$. Finally define $f\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $f(x) = g(x')$. Then by Proposition 5.5, there is $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that for infinitely many n we have that $f(x) = x_n$. Let $y = x'$.

If n is such that $f(x) = g(y) = x_n$, then as $x_n \leq_T \bar{x}^k = y_k, \forall k \leq n$, we have that $g(y) \leq_T y_k, \forall k \leq n$. Since this happens for infinitely many n, we have that $g(y) \leq_T y_n$, for all n. □

We do not know anything about $E_0^{\mathbb{N}}$ but if we let E_{ctble} be the equivalence relation $E_{ctble}^{2^{\mathbb{N}}}$ (so that $E_0^{\mathbb{N}} <_B E_{ctble}$), we have:

Proposition 5.8. *E_{ctble} fails (b) and (c).*

Proof. We will prove that E_{ctble} fails (b), the proof that it also fails (c) being similar. Let $X = (2^{\mathbb{N}})^{\mathbb{N}}, Y = 2^{\mathbb{N}}$, let μ be the usual product measure on Y and put $E = E_{ctble}$. Define $P \subseteq X \times Y$ by

$$(x, y) \in P \iff y \notin \{x_n : n \in \mathbb{N}\}.$$

Clearly $\mu(P_x) = 1$ and P is E-invariant. Assume now, towards a contradiction, that there is a Borel function $f\colon X \to Y^{\mathbb{N}}$ such that $\forall x \in X \forall n \in \mathbb{N}((x, f(x)_n) \in P)$ and $x_1 E x_2 \implies \{f(x_1)_n : n \in \mathbb{N}\} = \{f(x_2)_n : n \in \mathbb{N}\}$. Then

$$\forall x \in X\big(\{f(x)_n : n \in \mathbb{N}\} \cap \{x_n : n \in \mathbb{N}\} = \emptyset\big).$$

Define $F\colon X^{\mathbb{N}} \to Y^{\mathbb{N}}$ as follows: Fix a bijection $(i, j) \mapsto \langle i, j\rangle$ from $\mathbb{N}^2$ to $\mathbb{N}$ and for $n \in \mathbb{N}$ put $n = \langle n_0, n_1\rangle$. Given $x \in X^{\mathbb{N}}$, define $x' \in X$ by $x'_n = (x_{n_0})_{n_1}$. Then let $F(x) = f(x')$. First notice that for $x = (x_n), y = (y_n) \in X^{\mathbb{N}}$,

$$\{[x_n]_E : n \in \mathbb{N}\} = \{[y_n]_E : n \in \mathbb{N}\} \implies x' E y' \implies F(x) E F(y).$$

Thus by Theorem 5.6, there is some $x \in X^{\mathbb{N}}$ and $i \in \mathbb{N}$ such that $F(x) E x_i$, i.e., $f(x') E x_i$ or $\{f(x')_n : n \in \mathbb{N}\} = \{(x_i)_n : n \in \mathbb{N}\} = \{x'_{\langle i,n\rangle} : n \in \mathbb{N}\}$. Thus $\{f(x')_n : n \in \mathbb{N}\} \cap \{x'_n : n \in \mathbb{N}\} \neq \emptyset$, a contradiction. □

We do not know if E_{ctble} fails (d). We also do not know anything about equivalence relations induced by turbulent continuous actions of Polish groups on Polish spaces.

Finally, we note that by the dichotomy theorem of Hjorth concerning reducibility to countable (see [Hjo05] or [Kec24, Theorem 3.8]), in order to prove Conjecture 1.15 for Borel equivalence relations induced by Borel actions of Polish groups, it would be sufficient to prove it for Borel equivalence relations induced by stormy such actions.

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