This paper is the result of research last summer here at Caltech on a National Science Foundation grant. The paper is divided into two separate topics. The first shows the equivalence of several notions of measure uniformity for real sets. The second decomposes sets of Lebesgue measure zero to illustrate and investigate the properties of such a set. Since the two developments are not at all related, each is presented as a single unit.

This paper is a study of sets on the real line which possess certain "uniform" measure properties. It also should provide some insight into the character of the Lebesgue outer measure, especially as to its "smoothness".
0. Introduction. In this paper the concept of a cover tree for a subset of the real line is developed and used to investigate some properties of sets of Hebesque measure zero. This investigation shows that any set of Hebesque measure zero can be decomposed as the union of a $G_0$ set and a set which is dense in itself, and furthermore, any $G_0$ set of measure zero can be characterized as the union of a Generalized Cantor set and a countable set which contains no subsets which are dense in themselves.

If $E$ is a set of real numbers with finite Hebesque outer measure, denoted $m^*(E)$, then there exists a sequence of open sets $\{O^i_{j}\}_{j=1}^\infty$, with $E \subseteq O^i_j$ for all $j$, and $m^*(E) = \lim_{j \to \infty} m(O^i_j)$. We can require, further, that $0^1 \supseteq 0^2 \supseteq \cdots \supseteq 0^i \supseteq \cdots$. An open set can be written uniquely as a countable union of disjoint open intervals, and so

$O^i = \bigcup_{i=1}^{\infty} U^i_\cdot$. If, for any $i$ and $\hat{j}$, $E \cap U^i_{\hat{j}} = \emptyset$, we can remove that $U^i_{\hat{j}}$ from $O^i$ and renumber so that $U^i_\cdot \cap E = \emptyset$ for all $i$ and $\hat{j}$.

It follows that the set $\bigcup_{i=1}^{\infty} U^i_\cdot$ satisfies the following conditions:

1) $E \subseteq \bigcup_{i=1}^{\infty} U^i_\cdot$ for all $\hat{j}$,

2) $U^i_\cdot \cap U^{i'}_{\hat{j}} = \emptyset$ for all $i \neq \hat{i}$, and all $\hat{j}$,

3) $E \cap U^i_\cdot \neq \emptyset$ for all $i$ and $\hat{j}$,

4) For all $i$ and $\hat{j}$ there is an $k$ such that $U^i_{k} \in U^i_{\hat{j}}$ and

5) $m^*(E) = \lim_{j \to \infty} (\bigcup_{i=1}^{\infty} m(U^i_j))$. 
