This paper is the result of research last summer here at Caltech on a National Science Foundation grant. The paper is divided into two seperate topics. The first shows the equivalence of several notions of measure uniformity for real sets. The second decomposes sets of Lebesgue measure zero to illustrate and investigate the properties of such a set. Since the two developments are not at all related, each is presented as a single unit.

This paper is a study of sets on the real line which posses certain "uniform" measure properties. It also should provide some insight into the character of the Lebesque outer measure, especially as to its "smoothness".

0. Introduction. In this paper the concept of a cover tree for a subset of the real line is developed and used to investigate some properties of sets of Lebesgue measure zero. This investigation shows that any set of Lebesgue measure zero can be decomposed as the union of a G_δ set and a set which is dense in itself, and furthermore, any G_δ set of measure zero can be characterized as the union of a Generalized Cantor Set and a countable set which contains no subsets which are dense in themselves.

If E is a set of real numbers with finite Lebesgue outer measure, denoted $m^*(E)$, then there exists a sequence of open sets $\{O^j\}_{j=1}^{\infty}$, with $E \subseteq O^j$ for all j, and $m^*(E) = \lim_{j\to\infty} m(O^j)$. We can require, further, that $O^1 \supseteq O^2 \supseteq \cdots \supseteq O^j \supseteq \cdots$. An open set can be written uniquely as a countable union of disjoint open intervals, and so $O^j = \bigcup_{i=1}^{\infty} U_i^j$. If, for any i and j, $E \cap U_i^j = \emptyset$, we can remove that U_i^j from O^j and renumber so that $U_i^j \cap E \neq \emptyset$ for all i and j. It follows that the set $\{U_i^j\}_{i,j=1}^{\infty}$ satisfies the following conditions;

1) $E \subseteq \bigcup_{i=1}^{\infty} U_i^j$ for all j,

2) $U_i^j \cap U_k^j = \emptyset$ for all $i \neq k$, and all j,

3) $E \cap U_i^j \neq \emptyset$ for all i and j,

4) for all i and j there is a k such that $U_i^{j+1} \subseteq U_k^j$, and

5) $m^*(E) = \lim_{j\to\infty} \left(\sum_{i=1}^{\infty} m(U_i^j) \right)$.