I. INTRODUCTION

Frequently in linear algebra, problems arise in which one has a collection of, say, r orthonormal vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ with $r<n$, and one needs to find n-r additional vectors which form an orthonormal basis of $\mathbb{R}^n$. For example, when one finds the singular value decomposition of a matrix A of rank r, one finds certain singular vectors $\vec{v}_1, \ldots, \vec{v}_r$ which are orthonormal, and one needs a unitary matrix whose first r columns are $\vec{v}_1, \ldots, \vec{v}_r$. The Gram-Schmidt process alone is not sufficient to solve this problem, for it requires one to find n-r vectors which when taken with $\vec{v}_1, \ldots, \vec{v}_r$, are linearly independent. In other words, one would need a systematic procedure for choosing vectors which are linearly independent.

Consider the following problem which appears in [1]: if $\vec{x} = [x_1, \hat{x}]^t$, with $\vec{x}$ of unit length, $x_1 \neq -1$, x_1 real, then the matrix U, given by

$$U = \left[\begin{array}{c|c} x_1 & \hat{x}^* \\ \hline \hat{x} & \dfrac{\hat{x}\hat{x}^*}{1+x_1} - I \end{array}\right]$$

is unitary. Note that U has $\vec{x}$ as its first column. This suggests that it may be possible to solve the general problem of finding an orthogonal (or unitary) matrix whose first r columns are specified with a closed formula. This is what we will be concerned with in sections III and IV; sections V and VI will be concerned with algorithmic solutions to the problem.