ON THE MEAN VALUE OF THE MAGNITUDE OF AN EXPONENTIAL SUM INVOLVING THE DIVISOR FUNCTION

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1. Introduction

For a sequence $(a_n)_{n\geq 1}$ of arithmetical interest, it is often desirable in additive number theory to have, for large X, an understanding of the L^p norms of the exponential sum $M(\alpha) = \sum_{n\leq X} a_n e(n\alpha)$. A sufficiently good understanding of these can also lead to estimates for the measure of $\{\alpha \in [0,1] : |M(\alpha)| > \lambda\}$ for λ of an appropriate size.

The case of L^1 norms has recieved particular attention, and there are few tools to study them. For these problems, obtaining good lower bounds is particularly difficult in many cases as the main contribution to the L^1 norm is not dominated by large values on a small portion of the interval $[0,1]$. As a result, a large portion of the contribution may to the L^1 norm may end up coming from points far from those at which the exponential sum is easy to estimate (typically rationals with small denominator). We mention here some of the previous work on the topic.

Littlewood conjectured, and McGehee, Pigno, and Smith [11] proved that if S is some set of n integers, then $\int_0^1 |\sum_n \mathbb{1}_S(n)e(n\alpha)|d\alpha \gg \log n$.

For specific sequences, and in particular those for which a_n is related to the multiplicative structure of n, one expects the true value of the L^1 norm to be closer to the upper bound one obtains from Cauchy-Schwarz and Parseval. This is what happens in the case of sequences with elements chosed uniformly at random from $\{-1,1\}$ by Khintchine's inequality. Because of this, it is reasonable to expect that in many cases, if the coefficients have some multiplicative structure, then they should behave randomly with regards to additive considerations.

In the case that $a_n = \Lambda(n)$, the Von-Mangoldt function, it was shown by Vaughan [12] that $\int_0^1 |\sum_{n\leq X} \Lambda(n)e(n\alpha)|d\alpha \gg \sqrt{X}$, and it was shown by Goldston [6] that $\int_0^1 |\sum_{n\leq X} \Lambda(n)e(n\alpha)|d\alpha \leq \left(\frac{\sqrt{2}}{2} + o(1)\right)\sqrt{X\log X}$.

In the case that a_n is the indicator function for r-free integers, Balog and Ruzsa [4] showed that $\int_0^1 |\sum_{n\leq X} a_n e(n\alpha)|d\alpha \asymp X^{\frac{1}{r+1}}$, improving on work of Brüdern, Granville, Perelli, Vaughan, and Wooley [2].

In the case of the Möbius function, Balog and Ruzsa [4] have shown that $\int_0^1 |\sum_{n\leq X} \mu(n)e(n\alpha)|d\alpha \gg X^{1/6}$, improving on previous results of Balog and Perelli [3] and Balog and Ruzsa [5] who obtained the lower bounds $\gg \exp(\frac{c\log X}{\log\log X})$, $\gg X^{1/8-\varepsilon}$ respectively.

In the case of $\mathrm{GL}(1)$ objects, it is easy to show that if χ is some Dirichlet character modulo q, then $\int_0^1 |\sum_{n\leq X} \chi(n)e(n\alpha)|d\alpha \sim C_q \log X$ for some $C_q > 0$.

In this paper, we make a contribution to the $\mathrm{GL}(2)$ case by studying the divisor function $\tau(n) = \sum_{d|n} 1$. In particular, with

$$M(\alpha) = \sum_{n \leq X} \tau(n) e(n\alpha),$$

we prove the following result.

Theorem 1. *We have that*

$$\int_0^1 |M(\alpha)| = C\sqrt{X}\log X + O(\sqrt{X})$$

where

$$C = \frac{18}{\pi^3} - \frac{6\log 2}{\pi^3} - \frac{1}{4\pi} \approx 0.366\ldots.$$

This improves on the bound

$$\sqrt{X} \ll \int_0^1 |M(\alpha)| d\alpha \ll \sqrt{X}\log X.$$

obtained by Goldston and the author in [7]. This problem was previously attempted by Brüdern [1], who claimed to have shown that $\int_0^1 |M(\alpha)| \asymp \sqrt{X}$. It turned out that there was an error in the proof of a key lemma, which turned out to be false.

Let us now briefly describe the main ideas of the proof. We use Kloosterman's refinement of the circle method. This leads us to require good estimates on $M\left(\frac{a}{q} + \beta\right)$ for $q \leq \sqrt{X}, |\beta| \ll (q\sqrt{X})^{-2}$.

To do so, we must first replace the cutoff $\mathbb{1}_{[1,X]}(n)$ with $w\left(\frac{n}{X}\right)$ for w some smooth function supported on $[0,1]$ equal to 1 on $(\eta, 1-\eta)$ for some sufficiently small η. Due to the fact that the size of $\tau(n)$ may be large for various points, we require that η depends on X and is at most $o\left(\frac{1}{\log X}\right)$. The fact that we cannot suppose that the derivatives $w^{(j)}$ are $O_j(1)$ for all j requires us to be quite careful in some parts of the argument.

At this point, we use Voronoi summation to obtain that the exponential sum in question is equal to a relatively easily understood main term plus an error term roughly of the form $\frac{X}{q}\sum_n \tau(|n|) e\left(\frac{\bar{a}n}{q}\right) f_{q,\beta}(n)$ where $f_{q,\beta}$ is some reasonably well-understood function that can be thought of as (with the dependence on β largely coming from bounds on derivatives of w) being concentrated near the points $|n| \ll \frac{q^2}{X} \cdot X^{\varepsilon}$. We may bound this error term quite straightforwardly, using integration by parts to control the complications that arise if β is large, though even assuming that w does not depend on X, the bound one obtains is off from what we desire by about $\log^2 X$.

However, we may note that $f_{q,\beta}$ does not depend on a, so on average, as a ranges over coprime residue classes $\mod q$, there must be cancellation between summands in the error term for most a. This is expoited by Cauchy-Schwarz and orthogonality, giving a saving of $q^{1/2}$ for all somewhat large n which still contribute a nonnegligible amount, and further concentrates the relevant mass of f near 0, enough so that the potentially large size of τ at some points ceases to be a issue, removing the problematic factor of $\log^2 X$ by allowing us to treat the very first few terms differently so as to avoid the issues that come from smoothing. In the end, when $\beta = 0$, we obtain a bound of $O(X^{1/2})$ on average for the error term, which suffices.

Obtaining an asymptotic formula when $a_n = \lambda_f(n)$, the nth Fourier coefficient of some holomorphic cusp form f, seems significantly harder, as the exponential sum is small everywhere, and there is no "main term" around rationals with small denominators like we have in the case of the divisor function. Determining the L^1 norm up to a constant factor however, is quite easy. In particular, the estimate $\int_0^1 |\sum_{n\le X} \lambda_f(n)e(n\alpha)|d\alpha \asymp \sqrt{X}$ follows from Hölder's inequality, the asymptotic$\sum_{n\le X} |\lambda_f(n)|^2 \sim c_f X$, and the pointwise bound $\sum_{n\le X} \lambda_f(n)e(n\alpha) \ll \sqrt{X}$ of Jutila [8].

Another natural extension is to instead of $\tau(n)$, consider $\tau_3(n) = \sum_{d_1d_2d_3=n} 1$. The problem becomes significantly harder in this case however, since the effective length of the error term from Voronoi summation is significantly longer.

2. Setup

Let $Y = X^{1-\delta}$ with $\delta = \frac{1}{100}$, and let w be a smooth function taking values in $[0,1]$ supported on $[1/2, X]$ such that

$$w(u) = 1 \text{ for } 1 \le u \le X - Y$$
$$w^{(j)}(u) \ll Y^{-j} \text{ for } j \in \{0,1,2\}, u \ge 1$$

Instead of working with the L^1-norm of $M(\alpha)$, it suffices to work with the L^1-norm of

$$M^*(\alpha) = \sum_{n\le X} \tau(n)w(n)e(n\alpha)$$

since we have that by Parseval and Cauchy-Schwarz

$$\begin{aligned}
&\left| \int_0^1 |M(\alpha)|d\alpha - \int_0^1 |M^*(\alpha)|d\alpha \right| \\
&\le \int_0^1 \left| \sum_{X-Y<n\le X} |1-w(n)|\tau(n)e(n\alpha) \right| d\alpha \\
&\le \left(\int_0^1 \left| \sum_{X-Y<n\le X} |1-w(n)|\tau(n)e(n\alpha) \right|^2 d\alpha \right)^{1/2} \\
&\le \left(\sum_{X-Y<n\le X} \tau(n)^2 \right)^{1/2} \ll X^{1/2-\delta/2+\varepsilon}.
\end{aligned}$$

Let $Q = \sqrt{X}$. Then, we have by (20.9) and the proof of 20.7 in [10]

(2.1)
$$\int_0^1 |M^*(\alpha)|d\alpha = \sum_{q\le Q} \sum_{\substack{Q<a\le q+Q \\ (a,q)=1}} \int_{-1/(aq)}^0 \left| M^*\left(\frac{\overline{a}}{q} + \beta\right) \right| d\beta + \int_0^{1/(aq)} \left| M^*\left(-\frac{\overline{a}}{q} + \beta\right) \right| d\beta.$$

where $\overline{a}$ is so that $a\overline{a} \equiv 1 \pmod{q}$. From (4.49) in [10], we have that for all β and $q \le Q, (a,q) = 1$

$$M^*\left(\frac{a}{q}+\beta\right) = \sum_{n\ge 1}\tau(n)e\left(\frac{an}{q}\right)w(n)e(n\beta) = \frac{1}{q}\int_0^\infty(\log x + 2\gamma - 2\log q)w(x)e(x\beta)dx$$
$$+\sum_{n\in\mathbb{Z}}e\left(-\frac{\overline{a}n}{q}\right)\Delta(n,q,\beta)$$

where here

$$\Delta(n,q,\beta) = \begin{cases} -\frac{2\pi\tau(n)}{q}\int w(x)e(x\beta)Y_0\left(\frac{4\pi\sqrt{xn}}{q}\right)dx & n\ge 1\\ 0 & n=0\\ -\frac{4\tau(-n)}{q}\int w(x)e(x\beta)K_0\left(\frac{4\pi\sqrt{x|n|}}{q}\right)dx & n<0.\end{cases}$$

Here, Y_ν, K_ν are the standard Bessel functions. It follows that

$$\int_0^1 M^*(\alpha)d\alpha = E + R$$

where

$$E = \sum_{q\le Q}\sum_{\substack{Q<a\le q+Q\\(a,q)=1}}\int_{-1/(aq)}^{1/(aq)}|I_q(\beta)|d\beta$$

with

$$I_q(\beta) = \int_0^\infty(\log x + 2\gamma - 2\log q)w(x)e(x\beta)dx,$$

and

$$R \le \sum_{q\le Q}\sideset{}{^*}\sum_{a(q)}\int_{-1/(qQ)}^{1/(qQ)}|R_{a,q}(\beta)|d\beta$$

where

$$R_{a,q}(\beta) = \sum_n e\left(-\frac{\overline{a}n}{q}\right)\Delta(n,q,\beta).$$

Then, the main theorem follows from the following two lemmas, which we prove in the next two sections.

Lemma 2. *We have*

$$R \ll \sqrt{X}.$$

Lemma 3. *We have*

$$E = C\sqrt{X}\log X + O(\sqrt{X}).$$

3. The error term

In order to prove Lemma 2, we shall in fact show the following.

Lemma 4. *We have that for all $q \le Q, |\beta| \le 1/(qQ)$*

$$\sum_{a(q)}{}^{*}|R_{a,q}(\beta)| \ll q^2 + qX^{1/2} + X^{2\delta}q^{3/2+\varepsilon} + |\beta|^2 q^{5/2}X^{7/4} + |\beta|^2 q^{3/2+\varepsilon}X^{7/4+2\delta}.$$

Proof of Lemma 2 assuming Lemma 4. Clearly, we have that by Lemma 4

$$\begin{aligned}
R &= \sum_{q\le Q}\sum_{a(q)}{}^{*}\int_{-1/(qQ)}^{1/(qQ)}|R_{a,q}(\beta)|d\beta \\
&\ll \sum_{q\le Q}\int_{-1/(qQ)}^{1/(qQ)} q^2 + qX^{1/2} + X^{2\delta}q^{3/2+\varepsilon} + |\beta|^2 q^{5/2}X^{7/4} + |\beta|^2 q^{3/2+\varepsilon}X^{7/4+2\delta} d\beta \\
&\ll \sum_{q\le Q} 1 + \frac{q}{Q} + \frac{q^{1/2+\varepsilon}}{Q}X^{2\delta} + \frac{q^{-1/2}}{Q^3}X^{7/4} + \frac{q^{-3/2+\varepsilon}}{Q^3}X^{7/4+2\delta} \\
&\ll Q + Q^{1/2}X^{2\delta} + Q^{-5/2}X^{7/4} + X^{7/4-3/2+2\delta} \ll X^{1/2}
\end{aligned}$$

and the desired result follows. □

Proof of Lemma 4. Note that by Cauchy-Schwarz, we have

$$\begin{aligned}
R_q(\beta) &= \sum_{a(q)}{}^{*}\left|\sum_n e\left(-\frac{\bar{a}n}{q}\right)\Delta(n,q,\beta)\right| \\
&\le \sum_{m\in\mathbb{Z}}\sum_{a(q)}{}^{*}\left|\sum_{qm\le n<q(m+1)} e\left(-\frac{\bar{a}n}{q}\right)\Delta(n,q,\beta)\right| \\
&\le \sum_{m\in\mathbb{Z}} q^{1/2}\left(\sum_{a(q)}{}^{*}\left|\sum_{qm\le n<q(m+1)} e\left(-\frac{an}{q}\right)\Delta(n,q,\beta)\right|^2\right)^{1/2}
\end{aligned}$$

Let

$$B_{q,m}(\beta) = \sum_{a(q)}{}^{*}\left|\sum_{qm\le n<q(m+1)} e\left(-\frac{an}{q}\right)\Delta(n,q,\beta)\right|^2.$$

Note that then we have

$$\begin{aligned}
B_{q,m}(\beta) &= \sum_{a(q)}{}^{*}\left|\sum_{qm\le n<q(m+1)} e\left(-\frac{an}{q}\right)\Delta(n,q,\beta)\right|^2 \le \sum_{a(q)}\left|\sum_{qm\le n<q(m+1)} e\left(-\frac{an}{q}\right)\Delta(n,q,\beta)\right|^2 \\
&= \sum_{a(q)}\sum_{qm\le n_1,n_2<q(m+1)} e\left(-\frac{a(n_1-n_2)}{q}\right)\Delta(n_1,q,\beta)\overline{\Delta(n_2,q,\beta)} \\
&= \sum_{qm\le n_1,n_2<q(m+1)}\Delta(n_1,q,\beta)\overline{\Delta(n_2,q,\beta)}\sum_{a(q)} e\left(-\frac{a(n_1-n_2)}{q}\right) \\
&= q\sum_{qm\le n<q(m+1)}|\Delta(n,q,\beta)|^2
\end{aligned}$$

so it follows that

$$R_q(\beta) \leq \sum_{m\in\mathbb{Z}} q \left(\sum_{qm\leq n<q(m+1)} |\Delta(n,q,\beta)|^2 \right)^{1/2}. \tag{3.1}$$

Integrating by parts twice, and noting that w is supported on $[1/2, X]$ we obtain (with B_ν denoting either Y_ν or K_ν depending on the sign of n) that for all $n \neq 0$

$$\int_0^\infty e(x\beta)w(x)B_0\left(\frac{4\pi\sqrt{x|n|}}{q}\right)dx$$

$$= \frac{q^2}{4\pi^2|n|}\int_{1/2}^\infty e(x\beta)(w'' + 4\pi i\beta w' - 4\pi^2\beta^2 w)(x)\left(xB_2\left(\frac{4\pi\sqrt{x|n|}}{q}\right)\right)dx. \tag{3.2}$$

We have that from the bounds on w that

$$\begin{aligned}(w'' + 4\pi i\beta w' - 4\pi^2\beta^2 w)(x) &\ll X^{2\delta}(|\beta|^2 + (|\beta|x^{-1} + x^{-2})\mathbb{1}_{x\in[1/2,1]\cup[X-Y,X]}) \\ &\ll X^{2\delta}(|\beta|^2 + \mathbb{1}_{x\in[1/2,1]\cup[X-Y,X]}x^{-2}).\end{aligned} \tag{3.3}$$

so from (3.2) we obtain that for $|n| \gg q^2$, by (4.9) in [13]

$$\begin{aligned}&\int_0^\infty e(x\beta)w(x)B_0\left(\frac{4\pi\sqrt{x|n|}}{q}\right)dx \\ &\ll \frac{q^2}{|n|}\int_{1/2}^\infty X^{2\delta}(x|\beta|^2 + \mathbb{1}_{x\in[1/2,1]\cup[X-Y,X]}x^{-1})\left|B_2\left(\frac{4\pi\sqrt{x|n|}}{q}\right)\right|dx \\ &\ll \frac{q^2}{|n|}X^{2\delta}\left(\int_{1/2}^X x|\beta|^2 q^{1/2}x^{-1/4}|n|^{-1/4}dx + \int_{[1/2,1]\cup[X-Y,X]} x^{-1}q^{1/2}x^{-1/4}|n|^{-1/4}dx\right) \\ &\ll q^{5/2}|n|^{-5/4}|\beta|^2X^{7/4+2\delta} + q^{5/2}|n|^{-5/4}X^{2\delta}\end{aligned}$$

so by the divisor bound $\tau(n) \ll_\varepsilon n^\varepsilon$ we have that

$$\Delta(n,q,\beta) \ll |n|^\varepsilon q^{3/2}|n|^{-5/4}|\beta|^2X^{7/4+2\delta} + q^{3/2}|n|^{-5/4+\varepsilon}X^{2\delta}.$$

It follows that for $|m| > q$

$$\begin{aligned}\sum_{qm\leq n<q(m+1)} |\Delta(n,q,\beta)|^2 &\ll X^{4\delta}|m|^\varepsilon(q^4(q|m|)^{-5/2} + |\beta|^4q^4X^{7/2}(q|m|)^{-5/2}) \\ &\ll (q^{3/2} + |\beta|^4q^{3/2}X^{7/2})X^{4\delta}|m|^{-5/2+\varepsilon}\end{aligned}$$

so

$$\sum_{|m|>q}\left(\sum_{qm\leq n<q(m+1)}|\Delta(n,q,\beta)|^2\right)^{1/2} \ll X^{2\delta}(q^{1/2+\varepsilon} + |\beta|^2q^{1/2+\varepsilon}X^{7/4}). \tag{3.4}$$

From now on, until specified otherwise, we shall restrict ourselves to $|n| \leq q^2$, $|m| \ll q$.

For $n \neq 0$, by integration by parts

$$\Delta(n,q,\beta) \ll \frac{\tau(n)}{q}(|\Delta_1(n)| + |\Delta_2(n)| + |\Delta_3(n)|)$$

where

$$\Delta_1 = \Delta_1(n,q,\beta) = \frac{q\beta}{\sqrt{|n|}} \int_0^\infty e(x\beta) w(x) x^{1/2} B_1\left(\frac{4\pi\sqrt{x|n|}}{q}\right) dx$$

$$\Delta_2 = \Delta_2(n,q,\beta) = \frac{q}{\sqrt{|n|}} \int_{X-Y}^X e(x\beta) w'(x) x^{1/2} B_1\left(\frac{4\pi\sqrt{x|n|}}{q}\right) dx$$

$$\Delta_3 = \Delta_3(n,q,\beta) = \frac{q}{\sqrt{|n|}} \int_{1/2}^1 e(x\beta) w'(x) x^{1/2} B_1\left(\frac{4\pi\sqrt{x|n|}}{q}\right) dx.$$

We have, from (4.9) in [13] and integration by parts that

$$\begin{aligned}\Delta_1 &= \frac{q^2\beta}{|n|} \int (2\pi i\beta e(x\beta) w(x) + e(x\beta) w'(x)) \left(x B_2\left(\frac{4\pi\sqrt{x|n|}}{q}\right)\right) dx \\ &\ll q^2|\beta||n|^{-5/4}\Big(\int_{q^2/|n|}^X (|\beta| + |w'(x)|) q^{1/2} x^{3/4} dx + |n|^{-3/4} \int_{1/2}^{q^2/|n|} (|\beta| + |w'(x)|) q^2 dx \Big).\end{aligned}$$

The first integral is

$$\ll q^{1/2} X^{3/4} \int_{X-Y}^X |w'(x)| dx + q^{1/2}|\beta| X^{3/4} \int_1^X dx \ll q^{1/2}X^{3/4} + q^{1/2}X^{7/4}|\beta|$$

while the second one is

$$\ll |\beta| q^4 |n|^{-1} + q^2 \ll |\beta| q^{1/2} X^{7/4} + q^{1/2} X^{3/4}.$$

Combining these, we obtain that

$$\Delta_1 \ll |\beta| q^{5/2} X^{3/4} |n|^{-5/4} + |\beta|^2 q^{5/2} X^{7/4} |n|^{-5/4}. \tag{3.5}$$

Similarly, integrating by parts, we obtain that

$$\begin{aligned}\Delta_2 &\ll \frac{q^2}{|n|} \int_{X-Y}^X (|\beta||w'(x)| + |w''(x)|) x B_2\left(\frac{4\pi\sqrt{x|n|}}{q}\right) dx \\ &\ll \frac{q^2}{|n|} \int_{X-Y}^X (|\beta| Y^{-1} + X^{\delta-1} Y^{-1}) x B_2\left(\frac{4\pi\sqrt{x|n|}}{q}\right) dx \\ &\ll |\beta| q^{5/2} X^{3/4} |n|^{-5/4} + q^{5/2} X^{-1/4+\delta} |n|^{-5/4}. \end{aligned} \tag{3.6}$$

In addition, we have from (2.5) in [13] the bound $B_1(x) \ll x^{-1/2}$ for $x \gg 1$, so we obtain that

$$\Delta_2 \ll \frac{q}{\sqrt{|n|}} \int_{X-Y}^X (Y^{-1} q^{1/2} x^{-1/4}) x^{1/2} |n|^{-1/4} dx \ll q^{3/2} X^{1/4} |n|^{-3/4}. \tag{3.7}$$

Also, from the bound $B_1(y) \ll y^{-1}$, we obtain that

$$\Delta_3 \ll \frac{q^2}{|n|}. \tag{3.8}$$

Therefore, combining (3.5), (3.7), (3.6, and (3.8) we have that

$$|\Delta(n,q,\beta)| \ll \tau(|n|)\bigg(\frac{q}{|n|} + |\beta| q^{3/2} X^{3/4} |n|^{-5/4} + |\beta|^2 q^{3/2} X^{7/4} |n|^{-5/4} + \min(q^{3/2} X^{-1/4+\delta} |n|^{-5/4}, q^{1/2} X^{1/4} |n|^{-3/4})\bigg),$$

so we have that

$$\sum_{|n|\le 2q} |\Delta(n,q,\beta)|^2 \ll \sum_{0<n\le 2q} q^2 \frac{\tau(n)^2}{n^2} + |\beta|^2 q^3 X^{3/2} \frac{\tau(n)^2}{n^{5/2}} + |\beta|^4 q^3 X^{7/2} \frac{\tau(n)^2}{n^{5/2}} + qX^{1/2} \frac{\tau(n)^2}{n^{3/2}}$$

$$\ll q^2 + |\beta|^2 q^3 X^{3/2} + |\beta|^4 q^3 X^{7/2} + qX^{1/2}, \tag{3.9}$$

and for $1 < |m| \le q$

$$\sum_{qm \le n < q(m+1)} |\Delta(n,q,\beta)|^2$$
$$\ll q^{1+\varepsilon}\left(q^2 \cdot \frac{1}{|m|^2 q^2} + |\beta|^4 q^3 X^{3/2} \cdot \frac{1}{|m|^{5/2} q^{5/2}} + q^3 X^{-1/2+2\delta} \cdot \frac{1}{q^{5/2}|m|^{5/2}}\right)$$
$$\ll \frac{q^{1+\varepsilon}}{|m|^2} + \frac{|\beta|^4 q^{1/2} X^{3/2}}{|m|^{5/2}} + \frac{q^{3/2} X^{-1/2+2\delta}}{|m|^{5/2}}.$$

Therefore, it follows that

$$\sum_{1<|m|\le q} q\bigg(\sum_{qm\le n<q(m+1)} |\Delta(n,q,\beta)|^2\bigg)^{1/2}$$
$$\ll q \sum_{1<|m|\le q} q^{1/2+\varepsilon}\frac{1}{|m|} + |\beta|^2 q^{1/4+\varepsilon} X^{3/4} \frac{1}{|m|^{5/4}} + q^{3/4} X^{-1/4+\delta} \frac{1}{|m|^{5/4}}$$
$$\ll q^{3/2+\varepsilon} + |\beta|^2 q^{5/4+\varepsilon} X^{3/4} + q^{7/4} X^{-1/4+\delta}.$$

Putting this bound together with (3.9), (3.4), and (3.1) yields that

$$R_q(\beta) \ll q^2 + qX^{1/2} + |\beta|^2 q^{5/2} X^{7/4} + X^{2\delta} q^{3/2+\varepsilon} + |\beta|^2 q^{3/2+\varepsilon} X^{7/4+2\delta}. \tag{3.10}$$

The desired result follows. □

THE MAIN TERM

In this section, we prove Lemma 3. Note that we have that

$$E = \sum_{q\le Q} \sum_{\substack{Q<a\le q+Q \\ (a,q)=1}} \int_{-1/(aq)}^{1/(aq)} \frac{1}{q} |I_q(\beta)| d\beta = 2 \sum_{q\le Q} \frac{1}{q} \sum_{\substack{Q<a\le q+Q \\ (a,q)=1}} \int_0^{1/(aq)} |I_q(\beta)| d\beta$$

For $\beta < 1/X, q \leq Q$, we have that by the triangle inequality

$$\begin{aligned}|I_q(\beta)| &\leq \int_{1/2}^{X} |\log x + 2\gamma - 2\log q| dx \\ &\ll \int_{1/2}^{q^2} \log(q^2/x)dx + \int_{q^2}^{X} \log(x/q^2)dx + O(X) \\ &= q^2\log(q^2) - (q^2\log(q^2) - q^2) + X\log X - X - q^2\log q^2 + q^2 - (X - q^2)\log q^2 + O(X) \\ &= X\log(X/q^2) + O(X).\end{aligned}$$

It can then be checked that the contribution due to $\beta < 1/X$ can be disregarded. Indeed, we have that

$$\sum_{q\leq Q} \frac{\varphi(q)}{q} \int_0^{1/X} |I_q(\beta)| d\beta \leq \frac{1}{X} \sum_{q\leq Q} X(\log(X/q^2) + 1) \ll X^{1/2}.$$

We shall therefore now restrict our attention to $\beta \geq 1/X$. Note that we have that by integration by parts

$$I_q(\beta) = \int (\log x + 2\gamma - 2\log q) w(x) e(x\beta) d\beta = I_q^1(\beta) + I_q^2(\beta)$$

where

$$I_q^1(\beta) = \frac{1}{2\pi i\beta} \int \frac{w(x)}{x} e(x\beta) dx$$

$$I_q^2(\beta) = \frac{1}{2\pi i\beta} \int_{X-Y}^{X} (\log x + 2\gamma - 2\log q) w'(x) e(x\beta) dx.$$

We have that

$$I_q^2(\beta) \ll \frac{1}{\beta Y} \int_{X-Y}^{X} \log x + 2\gamma - 2\log q dx \ll \frac{1}{\beta}(\log(X/q^2) + 2\gamma).$$

Also, we have that

$$I_q^1(\beta) = \frac{1}{2\pi i\beta} \int_{1/2}^{\beta^{-1}} \frac{w(x)}{x} dx + O\left(\beta^{-1} \int_{1/2}^{\beta^{-1}} \frac{|e(x\beta) - 1|}{x} dx + \beta^{-1} \left| \int_{\beta^{-1}}^{X} \frac{w(x)}{x} e(x\beta) dx \right| \right).$$

From the inequality $|e(\alpha) - 1| \leq |\alpha|$, it follows that

$$\int_{1/2}^{\beta^{-1}} \frac{|e(x\beta) - 1|}{x} dx \leq 1$$

and by integration by parts, we have that for $\beta > 1/X$

$$\begin{aligned}\int_{\beta^{-1}}^{X} \frac{w(x)}{x} e(x\beta) dx &= \frac{1}{2\pi i} w(\beta^{-1}) + \frac{1}{2\pi i\beta} \int_{\beta^{-1}}^{X} \frac{w(x)}{x^2} e(x\beta) dx - \frac{1}{2\pi i\beta} \int_{X-Y}^{X} \frac{w'(x)}{x} e(x\beta) dx \\ &\ll 1 + \beta^{-1} \int_{\beta^{-1}}^{X} \frac{1}{x^2} dx + \beta^{-1} \int_{X-Y}^{X} Y^{-1} x^{-1} dx \ll 1.\end{aligned}$$

Therefore, we have that

$$I_q^1(\beta) = \frac{\log(\beta^{-1})}{2\pi i\beta} + O(\beta^{-1})$$

so

$$E = 2\sum_{q\le Q}\frac{1}{q}\sum_{\substack{Q<a\le q+Q\\(a,q)=1}}\int_{1/X}^{1/(aq)}|I_q(\beta)|d\beta$$
$$= \frac{1}{\pi}\sum_{q\le Q}\frac{1}{q}\sum_{\substack{Q<a\le q+Q\\(a,q)=1}}\frac{1}{q}\int_{1/X}^{1/(aq)}\frac{\log(\beta^{-1})}{\beta}d\beta + O\Big(\sum_{q\le Q}\frac{\varphi(q)}{q}\int_{1/X}^{1/(qQ)}\frac{1}{\beta}(\log(X/q^2)+1)d\beta\Big).$$

The term inside the $O(-)$ is

$$\ll \sum_{q\le Q}\log(X/q^2)^2+\log(X/q^2) \ll \sqrt{X}.$$

At this point, the lower bound of $\gg \sqrt{X}\log X$ follows. Indeed, we have

$$\sum_{q\le Q}\frac{1}{q}\sum_{\substack{Q<a\le q+Q\\(a,q)=1}}\frac{1}{q}\int_{1/X}^{1/(aq)}\frac{\log(\beta^{-1})}{2\pi\beta}d\beta \gg \sum_{q\le Q}\frac{\varphi(q)}{q}((\log X)^2-\log(2qQ)^2) \gg \sqrt{X}\log X$$

by partial summation.

Now we shall show that with

$$S = \frac{1}{\pi}\sum_{q\le Q}\frac{1}{q}\sum_{\substack{Q<a\le q+Q\\(a,q)=1}}\int_{1/X}^{1/(aq)}\frac{\log(\beta^{-1})}{\beta}d\beta$$
$$= \frac{1}{2\pi}\sum_{q\le Q}\frac{1}{q}\sum_{\substack{Q<a\le q+Q\\(a,q)=1}}\log^2 X-\log^2(aq),$$

$S = C\sqrt{X}\log X + O(\sqrt{X})$ with $C =$.

From the identity

$$\mathbb{1}_{(a,q)=1} = \sum_{\substack{d|a\\d|q}}\mu(d)$$

we obtain that

$$S = \sum_{q\le Q}\sum_{Q<a\le q+Q}(\log^2 X-\log(aq))\sum_{\substack{d|a\\d|q}}\mu(d) = S_2+S_3$$

where

$$S_2 = \frac{1}{2\pi}\sum_{Q\ge d>X^{1/4}}\mu(d)\sum_{\substack{q\le Q\\d|q}}\frac{1}{q}\sum_{\substack{Q<a\le q+Q\\d|a}}\log^2 X-\log^2(aq),$$

and

$$S_3 = \frac{1}{2\pi}\sum_{d\le X^{1/4}}\mu(d)\sum_{\substack{q\le Q\\d|q}}\frac{1}{q}\sum_{\substack{Q<a\le q+Q\\d|a}}\log^2 X-\log^2(aq).$$

Note that

$$
\begin{aligned}
S_2 &\ll \log^2 X \sum_{Q\ge d>X^{1/4}} \sum_{\substack{q\le Q\\ d|q}} \frac{1}{q} \sum_{\substack{Q<a\le q+Q\\ d|a}} 1 \\
&\ll \log^2 X \sum_{Q\ge d>X^{1/4}} \sum_{\substack{q\le Q\\ d|q}} \frac{1}{q}\cdot\frac{Q}{d} \\
&\ll Q\log^3 X \sum_{Q\ge d>X^{1/4}} \frac{1}{d^2} \\
&\ll X^{1/4}\log^3 X. \qquad (3.11)
\end{aligned}
$$

In addition, we have that

$$
\begin{aligned}
S_3 &= \frac{1}{2\pi}\sum_{d\le X^{1/4}} \mu(d) \sum_{\substack{q\le Q\\ d|q}} \frac{1}{q} \sum_{\substack{Q<a\le q+Q\\ d|a}} \log^2 X - \log^2(aq) \\
&= \frac{1}{2\pi}\sum_{d\le X^{1/4}} \mu(d) \sum_{q_0\le Q/d} \frac{1}{q_0 d} \sum_{Q/d<a_0\le q_0+Q/d} \log^2 X - \log^2(a_0 q_0 d^2) \\
&= \frac{1}{2\pi}\sum_{d\le X^{1/4}} \mu(d) \sum_{q_0\le Q/d} \frac{1}{q_0 d} \left(\int_{Q/d}^{q_0+Q/d} \log^2 X - \log^2(y q_0 d^2)dy + O(\log^2 X)\right) \\
&= \frac{1}{2\pi}\sum_{d\le X^{1/4}} \mu(d) \sum_{q_0\le Q/d} \frac{1}{q_0 d} \int_{Q/d}^{q_0+Q/d} \log^2 X - \log^2(y q_0 d^2)dy + O(-)
\end{aligned}
$$

where the error in the $O(-)$ is

$$
\ll \log^2 X \sum_{d\le X^{1/4}} \sum_{q_0\le Q/d} \frac{1}{q_0 d} \ll \log^3 X \sum_{d\le X^{1/4}} \frac{1}{d^2} \ll \log^3 X.
$$

The main term is equal to $S_4 - S_5$ with

$$
S_4 = \frac{1}{2\pi}\sum_{d\le X^{1/4}} \frac{\mu(d)}{d} \sum_{q_0\le Q/d} \log^2 X,
$$

$$
S_5 = \frac{1}{2\pi}\sum_{d\le X^{1/4}} \frac{\mu(d)}{d} \sum_{q_0\le Q/d} \frac{1}{q_0} \int_{Q/d}^{q_0+Q/d} \log^2(y q_0 d^2)dy.
$$

Note that

(3.12)

$$
S_4 = \frac{1}{2\pi} Q\log^2 X \sum_{d\le X^{1/4}} \frac{\mu(d)}{d^2}(1+O(X^{-1/4})) = \frac{1}{2\pi\zeta(2)}\sqrt{X}\log^2 X + O(X^{1/4}\log^2 X).
$$

In addition, we have that

$$
S_5 = \frac{Q}{2\pi}\sum_{d\le X^{1/4}} \frac{\mu(d)}{d^2} \sum_{q_0\le Q/d} \frac{1}{q_0} \int_{1}^{1+dq_0/Q} \log^2(y d q_0 Q)dy.
$$

With $A = dq_0Q$, the inner integral is equal to

$$\left[y(\log^2(Ay) - 2\log(Ay) + 2)\right]_1^{1+dq_0/Q}$$
$$= \left[y(\log^2 y + 2(\log A - 1)\log y + 2 - 2\log A + \log^2 A)\right]_1^{1+dq_0/Q}$$
$$= \frac{dq_0}{Q}(2 - 2\log A + \log^2 A) + \left(1 + \frac{dq_0}{Q}\right)\left(\log^2\left(1 + \frac{dq_0}{Q}\right) + 2(\log A - 1)\log\left(1 + \frac{dq_0}{Q}\right)\right)$$

so by various manipulations including the identities

$$\int_1^{Q/d} \log(1 + td/Q)\log(tdQ)\frac{dt}{t} = -Li_2(-1)\log X + O(1)$$

and

$$\frac{d}{Q}\int_1^{Q/d} \log(1 + td/Q)\log(tdQ)dt = (2\log 2 - 2)\log X + O(1)$$

where Li_s is the polylogarithm function, we obtain that

$$S_5 = \frac{Q}{2\pi}\sum_{d \le X^{1/4}} \frac{\mu(d)}{d^2}\int_1^{Q/d} \frac{d}{Q}(2 - 2\log(tdQ) + \log^2(tdQ))$$
$$+ \left(\frac{1}{t} + \frac{d}{Q}\right)\left(\log^2\left(1 + \frac{dt}{Q}\right) + 2(\log(tdQ) - 1)\log\left(1 + \frac{dt}{Q}\right)\right)dt + O(X^{1/4+\varepsilon})$$
$$= \frac{Q}{2\pi}\sum_{d \le X^{1/4}} \frac{\mu(d)}{d^2}\int_1^{Q/d} \frac{d}{Q}(-2\log(tdQ) + \log^2(tdQ))$$
$$+ \left(\frac{1}{t} + \frac{d}{Q}\right)\left(2\log(tdQ)\log\left(1 + \frac{dt}{Q}\right)\right)dt + O(X^{1/2})$$
$$= \frac{Q}{2\pi\zeta(2)}(\log^2 X - 4\log X + (2\log 2 - 2)\log X - \mathrm{Li}_2(-1)\log X) + O(X^{1/2})$$
$$= \frac{1}{2\pi\zeta(2)}\sqrt{X}\log^2 X - C\sqrt{X}\log X + O(X^{1/2})$$

where

$$C = \frac{3}{\pi^3}(4 + 2 - 2\log 2 + \mathrm{Li}_2(-1)) = \frac{18}{\pi^3} - \frac{6\log 2}{\pi^3} - \frac{1}{4\pi} \approx 0.366\ldots.$$

The desired result then follows.

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