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The Equivariant Tamagawa Number Conjecture: A survey

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(with an appendix by C. Greither)

Abstract. We give a survey of the equivariant Tamagawa number (a.k.a. Bloch-Kato) conjecture with particular emphasis on proven cases. The only new result is a proof of the 2-primary part of this conjecture for Tate-motives over abelian fields.

This article is an expanded version of a survey talk given at the conference on Stark's conjecture, Johns Hopkins University, Baltimore, August 5-9, 2002. We have tried to retain the succinctness of the talk when covering generalities but have considerably expanded the section on examples.

Most of the following recapitulates well known material due to many people. Section 3 is joint work with D. Burns (for which [14], [15], [16], [17] are the main references). In section 5.1 we have given a detailed proof of the main result which also covers the prime $l = 2$ (unavailable in the literature so far).

Contents

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Part 1. The Tamagawa Number Conjecture in the formulation of Fontaine and Perrin-Riou

The Tamagawa number conjecture of Bloch and Kato [10] is a beautiful generalization of the analytic class number formula (this is a theorem!) on the one hand, and the conjecture of Birch and Swinnerton- Dyer on the other. It was inspired by the computation of Tamagawa numbers of algebraic groups with, roughly speaking, motivic cohomology groups playing the role of commutative algebraic groups. In [35] and [34] Fontaine and Perrin-Riou found an equivalent formulation of the conjecture which has two advantages over the original one: It applies to any integer argument of the L-function (rather than just those corresponding to motives of negative weight), and it generalizes to motives with coefficients in an algebra other than $\mathbb{Q}$. Independently, Kato developed similar ideas in [48] and [49]. In this section we sketch this formulation.

1. The setup

Suppose given a smooth projective variety

$$X \to \mathrm{Spec}(\mathbb{Q})$$

and integers $i, j \in \mathbb{Z}$. The "motive" $M = h^i(X)(j)$ is the key object to which both an L-function and all the data conjecturally describing the leading coefficient of this function are attached. For the purpose of discussing L-functions, one need not appeal to any more elaborate notion of motive than that which identifies M with this collection of data (the "realisations" and the "motivic cohomology" of M). One has

- $M_l = H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)(j)$ a continuous representation of the Galois group $G_{\mathbb{Q}}$.
- The characteristic polynomial $P_p(T) = \det(1 - \mathrm{Fr}_p^{-1} \cdot T | M_l^{I_p}) \in \mathbb{Q}_l[T]$ where $\mathrm{Fr}_p \in G_{\mathbb{Q}}$ is a Frobenius element. It is conjectured, and known if X has good reduction at p, that $P_p(T)$ lies in $\mathbb{Q}[T]$ and is independent of l.
- The L-function $L(M, s) = \prod_p P_p(p^{-s})^{-1}$, defined and analytic for $\Re(s)$ large enough.
- The Taylor expansion
$$L(M, s) = L^*(M)s^{r(M)} + \cdots$$
at $s = 0$. That $L(M, s)$ can be meromorphically continued to $s = 0$ is part of the conjectural framework. This continuation is known, for example, if X is of dimension 0, or X is an elliptic curve [13] or a Fermat curve $x^N + y^N = z^N$ [76] and i and j are arbitrary.

Aim: Describe $L^*(M) \in \mathbb{R}^\times$ and $r(M) \in \mathbb{Z}$.

Examples. a) If $M = h^0(\mathrm{Spec}\, L)(0)$ for a number field L then $L(M, s)$ coincides with the Dedekind Zeta function $\zeta_L(s)$. If we write $L \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ then $r(M) = r_1 + r_2 - 1$ and $L^*(M) = -hR/w$ (R the unit regulator of L, w the number of roots of unity in L and h the class number of $\mathcal{O}_L$). This is the analytic class number formula.

b) If $M = h^1(A)(1)$ for an abelian variety A over a number field (or in fact for any X as above), then $L(M, s-1)$ is the classical Hasse-Weil L-function of the dual abelian variety $\check{A}$ (or the Picard variety $\mathrm{Pic}^0(X)$ of X). Since A and $\check{A}$ are isogenous, $L(M, s-1)$ also coincides with the Hasse-Weil L-function of A.

The *weight* of M is the integer $i - 2j$.

2. Periods and Regulators

We have four $\mathbb{Q}$-vector spaces attached to M.

- A finite dimensional space $M_B = H^i(X(\mathbb{C}), \mathbb{Q})(j)$ which carries an action of complex conjugation and a Hodge structure (see [27] for more details).
- A finite dimensional filtered space $M_{dR} = H^i_{dR}(X/\mathbb{Q})(j)$.
- Motivic cohomology spaces $H^0_f(M)$ and $H^1_f(M)$ which may be defined in terms of algebraic K-theory. For example, if X has a regular, proper flat model $\mathfrak{X}$ over $\mathrm{Spec}(\mathbb{Z})$ and $i - 2j \neq -1$ then

$$H^0_f(M) = \mathrm{CH}^j(X) \otimes \mathbb{Q}/\text{hom. equiv.} \quad \text{if } M = h^{2j}(X)(j)$$

$$H^1_f(M) = \mathrm{im}\left((K_{2j-i-1}(\mathfrak{X}) \otimes \mathbb{Q})^{(j)} \to (K_{2j-i-1}(X) \otimes \mathbb{Q})^{(j)} \right).$$

Using alterations this image space can also be defined without assuming the existence of a regular model $\mathfrak{X}$ [69].

The spaces $H^0_f(M)$ and $H^1_f(M)$ are conjectured to be finite dimensional but essentially the only examples where this is known are those mentioned above:

Examples continued. a) For $M = h^0(\mathrm{Spec}(L))$ we have $H^0_f(M) = \mathbb{Q}$ and $H^1_f(M) = 0$ whereas for $M = h^0(\mathrm{Spec}(L))(1)$ we have $H^0_f(M) = 0$ and $H^1_f(M) = \mathcal{O}_L^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. For $M = h^0(\mathrm{Spec}(L))(j)$ it is also known that $H^1_f(M) = K_{2j-1}(L) \otimes \mathbb{Q}$ is finite dimensional [12].

b) For $M = h^1(X)(1)$ we have $H^0_f(M) = 0$ and $H^1_f(M) = \mathrm{Pic}^0(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. This last space is finite dimensional by the Mordell-Weil theorem.

For a $\mathbb{Q}$-vector space W and a $\mathbb{Q}$-algebra R we put $W_R = W \otimes_{\mathbb{Q}} R$. The period isomorphism $M_{B,\mathbb{C}} \cong M_{dR,\mathbb{C}}$ induces a map

$$\alpha_M : M^+_{B,\mathbb{R}} \to (M_{dR}/\,\mathrm{Fil}^0\, M_{dR})_{\mathbb{R}}.$$

For any motive M one has a dual motive M^* with dual realizations. For example, if $M = h^i(X)(j)$ where X is of dimension d then Poincare duality gives a perfect pairing

$$H^i(X)(j) \times H^{2d-i}(X)(d-j) \to H^{2d}(X)(d) \xrightarrow{\mathrm{tr}} \mathbb{Q}$$

which identifies M^* with $h^{2d-i}(X)(d-j)$.

Conjecture $\mathbf{Mot}_\infty$: There exists an exact sequence

$$0 \to H^0_f(M)_{\mathbb{R}} \xrightarrow{c} \ker(\alpha_M) \to H^1_f(M^*(1))^*_{\mathbb{R}} \xrightarrow{h} H^1_f(M)_{\mathbb{R}} \xrightarrow{r} \mathrm{coker}(\alpha_M) \to H^0_f(M^*(1))^*_{\mathbb{R}} \to 0$$

Here c is a cycle class map, h a height pairing, and r the Beilinson regulator. Again, the exactness of this sequence is only known in a few cases, essentially those given by our standard examples a) and b).

Conjecture 1 (Vanishing Order):

$$r(M) = \dim_{\mathbb{Q}} H^1_f(M^*(1)) - \dim_{\mathbb{Q}} H^0_f(M^*(1))$$

Remark. The appearance of the dual motive $M^*(1)$ in the last two conjectures can be understood in two steps. First, one may conjecture that there are groups $H^i_c(M)$ ("motivic cohomology with compact support and $\mathbb{R}$-coefficients") which fit into a long exact sequence

$$\cdots \to H^i_c(M) \to H^i_f(M) \to H^i_{\mathcal{D}}(\mathbb{R}, M) \to \cdots$$

and where $H^0_{\mathcal{D}}(\mathbb{R}, M) \cong \ker(\alpha_M)$, $H^1_{\mathcal{D}}(\mathbb{R}, M) \cong \operatorname{coker}(\alpha_M)$ (a definition of such groups, Arakelov Chow groups in his terminology, has been given by Goncharov in [40]). Secondly, one may conjecture a perfect duality of finite dimensional $\mathbb{R}$-vector spaces

$$H^i_c(M) \times H^{2-i}_f(M^*(1)) \to H^2_c(\mathbb{Q}(1)) \cong \mathbb{R},$$

an archimedean analogue of Poitou-Tate duality. Then Conjecture 1 says that $r(M)$ is the Euler characteristic of motivic cohomology with compact support.

Define a $\mathbb{Q}$-vector space of dimension 1

$$\begin{aligned}\Xi(M) := &\operatorname{Det}_{\mathbb{Q}}(H^0_f(M)) \otimes \operatorname{Det}^{-1}_{\mathbb{Q}}(H^1_f(M))\\ &\otimes \operatorname{Det}_{\mathbb{Q}}(H^1_f(M^*(1))^*) \otimes \operatorname{Det}^{-1}_{\mathbb{Q}}(H^0_f(M^*(1))^*)\\ &\otimes \operatorname{Det}^{-1}_{\mathbb{Q}}(M^+_B) \otimes \operatorname{Det}_{\mathbb{Q}}(M_{dr}/\operatorname{Fil}^0)\end{aligned}$$

The exact sequence in Conjecture Mot_∞ induces an isomorphism

$$\vartheta_\infty : \mathbb{R} \cong \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}$$

Conjecture 2 (Rationality):

$$\vartheta_\infty(L^*(M)^{-1}) \in \Xi(M) \otimes 1$$

This conjecture goes back to Deligne [27][Conj. 1.8] in the critical case (i.e. where α_M is an isomorphism) and Beilinson [5] in the general case.

3. Galois cohomology

Throughout this section we refer to [16] for unexplained notation and further details. Define for each prime p a complex $R\Gamma_f(\mathbb{Q}_p, M_l)$

$$= \begin{cases} M_l^{I_p} \xrightarrow{1-\mathrm{Fr}_p} M_l^{I_p} & l \neq p \\ D_{cris}(M_l) \xrightarrow{(1-\mathrm{Fr}_p, \pi)} D_{cris}(M_l) \oplus D_{dR}(M_l)/\operatorname{Fil}^0 & l = p \end{cases}$$

One can construct a map of complexes $R\Gamma_f(\mathbb{Q}_p, M_l) \to R\Gamma(\mathbb{Q}_p, M_l)$ and one defines $R\Gamma_{/f}(\mathbb{Q}_p, M_l)$ as the mapping cone so that there is a distinguished triangle

$$R\Gamma_f(\mathbb{Q}_p, M_l) \to R\Gamma(\mathbb{Q}_p, M_l) \to R\Gamma_{/f}(\mathbb{Q}_p, M_l)$$

in the derived category of $\mathbb{Q}_l$ -vector spaces.

Let S be a finite set of primes containing l, ∞ and primes of bad reduction. There are distinguished triangles

$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) \to R\Gamma(\mathbb{Z}[\frac{1}{S}], M_l) \to \bigoplus_{p\in S} R\Gamma(\mathbb{Q}_p, M_l)$$

$$R\Gamma_f(\mathbb{Q}, M_l) \to R\Gamma(\mathbb{Z}[\frac{1}{S}], M_l) \to \bigoplus_{p\in S} R\Gamma_{/f}(\mathbb{Q}_p, M_l)$$

$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) \to R\Gamma_f(\mathbb{Q}, M_l) \to \bigoplus_{p\in S} R\Gamma_f(\mathbb{Q}_p, M_l) \tag{3.1}$$

Conjecture Mot$_l$: There are natural isomorphisms $H^0_f(M)_{\mathbb{Q}_l} \cong H^0_f(\mathbb{Q}, M_l)$ (cycle class map) and $H^1_f(M)_{\mathbb{Q}_l} \cong H^1_f(\mathbb{Q}, M_l)$ (Chern class map).

One can construct an isomorphism $H^i_f(\mathbb{Q}, M_l) \cong H^{3-i}_f(\mathbb{Q}, M_l^*(1))^*$ for all i. Hence Conjecture Mot$_l$ computes the cohomology of $R\Gamma_f(\mathbb{Q}, M_l)$ in all degrees.

The exact triangle (3.1) induces an isomorphism

$$\vartheta_l : \Xi(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong \mathrm{Det}_{\mathbb{Q}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l)$$

Let $T_l \subset M_l$ be any $G_{\mathbb{Q}}$-stable $\mathbb{Z}_l$-lattice.

Conjecture 3 (Integrality):

$$\mathbb{Z}_l \cdot \vartheta_l \vartheta_\infty (L^*(M)^{-1}) = \mathrm{Det}_{\mathbb{Z}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)$$

This conjecture (for all l) determines $L^*(M) \in \mathbb{R}^\times$ up to sign. It assumes Conjecture 2 and is independent of the choice of S and T_l [16][Lemma 5]. For $M = h^0(\mathrm{Spec}(L))$ (resp. $M = h^1(X)(1)$) it is equivalent to the l-primary part of the analytic class number formula (resp. Birch and Swinnerton-Dyer conjecture). For a sketch of the argument giving this equivalence we refer to section 5.4 below.

Part 2. The Equivariant Refinement

In many situations one has 'extra symmetries', more precisely there is a semisimple, finite dimensional $\mathbb{Q}$-algebra A acting on M.

Examples:

- X an abelian variety, $A = \mathrm{End}(X) \otimes \mathbb{Q}$
- $X = X' \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(K)$, $K/\mathbb{Q}$ Galois with group G, $A = \mathbb{Q}[G]$
- X a modular curve, A the Hecke algebra

4. Commutative Coefficients

If A is **commutative** (i.e. a product of number fields) one can construct $L({}_AM,s)$, $\Xi({}_AM)$, ${}_A\vartheta_\infty$, ${}_A\vartheta_l$ as before using determinants over A, $A\otimes\mathbb{R}$, $A\otimes\mathbb{Q}_l$. $L({}_AM,s)$ is a meromorphic function with values in $A\otimes\mathbb{C}$ and

$$\begin{aligned} r({}_AM) \in& H^0(\mathrm{Spec}(A\otimes\mathbb{R}),\mathbb{Z}) \\ L^*({}_AM) \in& (A\otimes\mathbb{R})^\times. \end{aligned}$$

For a finitely generated A-module P we denote by $\dim_A P$ the function $\mathfrak{p}\mapsto \mathrm{rank}_{A_\mathfrak{p}}(P_\mathfrak{p})$ on $\mathrm{Spec}(A)$.

One gets refinements of Conjectures 1 and 2 in a straightforward way.

Conjecture 1 (Equivariant Version):

$$r({}_AM) = \dim_A H^1_f(M^*(1)) - \dim_A H^0_f(M^*(1))$$

Conjecture 2 (Equivariant Version):

$${}_A\vartheta_\infty(L^*({}_AM)^{-1}) \in \Xi({}_AM)\otimes 1$$

Somewhat more interesting is the generalization of Conjecture 3. There are many $\mathbb{Z}$-orders $\mathfrak{A}\subseteq A$ unlike in the case $A=\mathbb{Q}$. It turns out that in order to formulate a conjecture over $\mathfrak{A}$ one additional assumption is necessary.

Assume that there is a **projective** $G_\mathbb{Q}$-stable $\mathfrak{A}_l := \mathfrak{A}\otimes\mathbb{Z}_l$ lattice $T_l\subset M_l$.

Then $R\Gamma_c(\mathbb{Z}[\frac{1}{S}],T_l)$ is a perfect complex of $\mathfrak{A}_l$-modules and $\mathrm{Det}_{\mathfrak{A}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}],T_l)$ is an invertible $\mathfrak{A}_l$-module. Since $\mathfrak{A}_l$ is a product of local rings this means that $\mathrm{Det}_{\mathfrak{A}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}],T_l)$ is in fact free of rank 1 over $\mathfrak{A}_l$. The existence of a projective lattice is guaranteed if $\mathfrak{A}$ is a maximal $\mathbb{Z}$-order in A. If $M = M_0\otimes h^0(\mathrm{Spec}(L))$ arises by base change of a motive M_0 to a finite Galois extension $L/\mathbb{Q}$ with group G then there is a projective lattice over the order $\mathfrak{A}=\mathbb{Z}[G]$ in $A=\mathbb{Q}[G]$.

Conjecture 3 (Equivariant Version):

$$\mathfrak{A}_l\cdot {}_A\vartheta_l\left({}_A\vartheta_\infty(L^*({}_AM)^{-1})\right) = \mathrm{Det}_{\mathfrak{A}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}],T_l)$$

This conjecture (for all l) determines $L^*({}_AM)\in(A\otimes\mathbb{R})^\times$ up to $\mathfrak{A}^\times$. Taking "Norms form A to $\mathbb{Q}$" one deduces the original Conjectures 1-3 from their equivariant refinements [16][Remark 11].

5. Proven cases

If $\mathfrak{A}$ is a maximal order in A, in particular if $\mathfrak{A}=\mathbb{Z}$, then Conjecture 3 has been considered traditionally in various cases, most notably in our Examples a) and b) above. In this section we review proven cases of Conjecture 3 with particular emphasis on non-maximal orders $\mathfrak{A}$. One should note here that Conjecture 3 over an order $\mathfrak{A}$ implies Conjecture 3 over any larger order $\mathfrak{A}'\supseteq\mathfrak{A}$ but not vice versa.

5.1. Abelian extensions of $\mathbb{Q}$ and the main conjecture of Iwasawa Theory. The following theorem summarizes what is known about our Example a) when $L/\mathbb{Q}$ is abelian.

Theorem 5.1. *(Burns-Greither) Let $L/\mathbb{Q}$ be a Galois extension with abelian group G, $M = h^0(\mathrm{Spec}(L))(j)$ with $j \in \mathbb{Z}$, $\mathfrak{A} = \mathbb{Z}[G]$ and l any prime number. Then Conjecture 3 holds.*

Corollary 5.1. *For any abelian field L and prime number l the cohomological Lichtenbaum conjecture (see [45][Thm. 1.4.1]) holds for the Dedekind Zeta-function at any $j \in \mathbb{Z}$.*

Proof. The cohomological Lichtenbaum conjecture is a rather immediate reformulation of Conjecture 3 for $M = h^0(\mathrm{Spec}(L))(j)$ and $\mathfrak{A} = \mathbb{Z}$ (see the proof of [45][Thm. 1.4.1]). Hence it follows by general functoriality [16][Remark 11] from Theorem 5.1. □

We shall give the proof of Theorem 5.1 for $j \leq 0$ in some detail in order to demonstrate how the formalism above unfolds in a concrete situation. One may also expect that the way in which we use the Iwasawa main conjecture and results on l-adic L-functions will be fairly typical for proofs of Conjecture 3 in a number of other situations. In essence we follow the proof [18] by Burns and Greither but our arguments cover the case $l = 2$ whereas those of [18] do not. We shall deduce Conjecture 3 for $j \leq 0$ from an Iwasawa theoretic statement (Theorem 5.2 below). This descent argument is fairly direct except for difficulties arising from trivial zeros of the l-adic L-function for $j = 0$. These can be overcome by using the Theorem of Ferrero-Greenberg [33] for odd χ and results of Solomon [71] for even χ. The main difficulty for $j < 0$ (identification of the image of Beilinson's elements in $K_{1-2j}(L)$ under the étale Chern class map) has already been dealt with by Huber and Wildeshaus in [46]. Such a proof of Conjecture 3 by descent from Theorem 5.2 is also possible for $j \geq 1$ provided one knows the non-vanishing of the l-adic L-function at j. Since this is currently the case only for $j = 1$ (as a consequence of Leopoldts conjecture for abelian fields [75][Cor. 5.30]) we do not give the details of this line of argument. Suffice it to say that Theorem 5.1 for $j \geq 1$ is then proven somewhat indirectly by appealing to compatibility of Conjecture 3 with the functional equation of the L-function (see the recent preprint [6]). We do not address the issue of compatibility with the functional equation in this survey. We also do not go into the proof of the Iwasawa main conjecture because it is by now fairly well documented in the literature, even for $l = 2$ (see [41] for a proof via Euler systems).

Finally we remark that another proof of Theorem 5.1 (but only for $\mathfrak{A}$ a maximal order and $l \neq 2$) has been given by Huber and Kings in [45]. This proof also appeals to compatibility with the functional equation. Theorem 5.1 for $j < 0$ and $\mathfrak{A} = \mathbb{Z}$ was proven before by Kolster, Nguyen Quang-Do and Fleckinger [54] (with final corrections in [7]).

We need some notation. For an integer $m \geq 1$ let $\zeta_m = e^{2\pi i/m}$, $L_m = \mathbb{Q}(\zeta_m)$, $\sigma_m : L_m \to \mathbb{C}$ the inclusion (which we also view as an archimedean place of L_m) and $G_m = \mathrm{Gal}(L_m/\mathbb{Q})$. By the Kronecker-Weber Theorem and general functoriality [16][Prop. 4.1b)] it suffices to prove Conjecture 3 for L_m in order to deduce it for all abelian $L/\mathbb{Q}$. By the same argument we may, and occasionally will assume that

m has at least two distinct prime factors. Let $\hat{G}_m$ be the set of complex characters of G_m, for $\eta \in \hat{G}_m$ let $e_\eta \in \mathbb{C}[G_m]$ be the idempotent $|G_m|^{-1}\sum_{g\in G_m}\eta(g)g^{-1}$ and denote by $L(\eta, s)$ the Dirichlet L-function of η.

Explicit formulas for Dirichlet L-functions. In this section we fix $1 < m \not\equiv 2$ mod 4 and write $M = h^0(\mathrm{Spec}(L_m))$, $A = \mathbb{Q}[G_m]$ and $\mathfrak{A} = \mathbb{Z}[G_m]$. One has

$$L({}_AM, s) = (L(\eta, s))_{\eta\in\hat{G}_m} \in \prod_{\eta\in\hat{G}_m} \mathbb{C} = A \otimes \mathbb{C}$$

and for $M = h^0(\mathrm{Spec}(L_m))$ the sequence in Conjecture $\mathbf{Mot}_\infty$ is the $\mathbb{R}$-dual (with contragredient G_m-action) of the unit regulator sequence

$$0 \to \mathcal{O}_{L_m}^\times \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{-\log|\ \ |_v} \bigoplus_{v|\infty} \mathbb{R} \xrightarrow{\Sigma} \mathbb{R} \to 0.$$

So one has

$$\Xi({}_AM)^\# = \mathrm{Det}_A^{-1}(\mathcal{O}_{L_m}^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes \mathrm{Det}_A(X_{\{v|\infty\}} \otimes_{\mathbb{Z}} \mathbb{Q})$$

where for any set S of places of L_m we define $Y_S = Y_S(L_m) := \bigoplus_{v\in S}\mathbb{Z}$ and $X_S = X_S(L_m)$ to be the kernel of the sum map $Y_S \to \mathbb{Z}$. Moreover, the superscript $\#$ indicates that the G_m-action has been twisted by the automorphism $g \mapsto g^{-1}$ of G_m.

It is well known that

$$L(\eta, 0) = -\sum_{a=1}^{f_\eta}\left(\frac{a}{f_\eta} - \frac{1}{2}\right)\eta(a)$$

$$\frac{d}{ds}L(\eta, s)|_{s=0} = -\frac{1}{2}\sum_{a=1}^{f_\eta}\log|1 - e^{2\pi ia/f_\eta}|\eta(a) \quad \eta \neq 1 \text{ even}$$

where $f_\eta | m$ is the conductor of η and $\eta(a) = 0$ for $(a, f_\eta) > 1$, and that $L(\eta, 0) \neq 0$ if and only if $\eta = 1$ or η is odd [75][Ch. 4]. One deduces Conjecture 1 for $M = h^0(\mathrm{Spec}(L_m))$ since

$$\dim_{\mathbb{C}} e_\eta(\mathcal{O}_{L_m}^\times \otimes_{\mathbb{Z}} \mathbb{C}) = \begin{cases} 1 & \eta \neq 1 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Before dealing with Conjecture 2 we introduce some further notation. We denote by $\hat{G}_m^{\mathbb{Q}}$ the set of $\mathbb{Q}$-rational characters , i.e. $\mathrm{Aut}(\mathbb{C})$-orbits on $\hat{G}_m$. For each $\chi \in \hat{G}_m^{\mathbb{Q}}$ we put $e_\chi = \sum_{\eta\in\chi} e_\eta \in A$ and we denote by $\mathbb{Q}(\chi)$ the field generated by the values of η for any $\eta \in \chi$. Then there is a ring isomorphism

$$A = \prod_{\chi\in\hat{G}_m^{\mathbb{Q}}} \mathbb{Q}(\chi).$$

We put

$$L(\chi, 0) := \sum_{\eta\in\chi} L(\eta, 0)e_\eta \in Ae_\chi \cong \mathbb{Q}(\chi)$$

and note that $L(\chi, 0)^\# := \sum_{\eta\in\chi} L(\eta^{-1}, 0)e_\eta$.

For $f_\eta \neq 1$ the image of $(1 - \zeta_{f_\eta}) \in L_m$ under the regulator map is

$$(1 - \zeta_{f_\eta}) \mapsto -\sum_{a\in(\mathbb{Z}/m\mathbb{Z})^\times/\pm 1} \log|1 - e^{2\pi ia/f_\eta}|^2 \tau_a^{-1}(\sigma_m)$$

where τ_a is the automorphism $\zeta_m \mapsto \zeta_m^a$ of L_m, and hence for any even character $\eta \neq 1$ the image of $e_\eta(1-\zeta_{f_\eta})$ is $L'(\eta^{-1},0)\cdot 2\cdot[L_m:L_{f_\eta}]\cdot e_\eta(\sigma_m)$. Note here that $e_\eta\tau_a = \eta(a)e_\eta$, and that σ_m (resp. $1-\zeta_{f_\eta}$ if f_η is a prime power) lies in the larger $\mathfrak{A}$-module $Y_{\{v|\infty\}} \supset X_{\{v|\infty\}}$ (resp. $\mathcal{O}_{L_{f_\eta}}[\frac{1}{f_\eta}]^\times \supset \mathcal{O}_{L_{f_\eta}}^\times$) but application of e_η (or equivalently $-\otimes_{\mathfrak{A}}\mathbb{Q}(\eta)$) turns this inclusion into an equality for $\eta \neq 1$.

There is a canonical isomorphism

$$\begin{aligned}
&\Xi({}_AM)^\# \xrightarrow{\sim} \operatorname{Det}_A^{-1}(\mathcal{O}_{L_m}^\times \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \otimes \operatorname{Det}_A(X_{\{v|\infty\}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \\
\xrightarrow{\sim} &\prod_{\substack{\chi\neq 1\\ \text{even}}} \operatorname{Det}_{\mathbb{Q}(\chi)}^{-1}(\mathcal{O}_{L_m}^\times \underset{\mathfrak{A}}{\otimes} \mathbb{Q}(\chi)) \otimes \operatorname{Det}_{\mathbb{Q}(\chi)}(X_{\{v|\infty\}} \underset{\mathfrak{A}}{\otimes} \mathbb{Q}(\chi)) \times \prod_{\text{other } \chi} \mathbb{Q}(\chi) \\
\xrightarrow{\sim} &\prod_{\substack{\chi\neq 1\\ \text{even}}} (\mathcal{O}_{L_m}^\times \underset{\mathfrak{A}}{\otimes} \mathbb{Q}(\chi))^{-1} \otimes_{\mathbb{Q}(\chi)} (X_{\{v|\infty\}} \underset{\mathfrak{A}}{\otimes} \mathbb{Q}(\chi)) \times \prod_{\text{other } \chi} \mathbb{Q}(\chi)
\end{aligned}$$

and in this description ${}_A\vartheta_\infty(L^*({}_AM,0)^{-1}) = (L^*({}_AM,0)^{-1})^\# {}_A\vartheta_\infty(1)$ has components

$$ {}_A\vartheta_\infty(L^*({}_AM,0)^{-1})_\chi = \begin{cases} 2\cdot[L_m:L_{f_\chi}][1-\zeta_{f_\chi}]^{-1}\otimes\sigma_m & \chi\neq 1 \text{ even} \\ \left(L(\chi,0)^\#\right)^{-1} & \text{otherwise.}\end{cases} \tag{5.1}$$

We now fix a prime number l and put $A_l := A\otimes\mathbb{Q}_l$. The isomorphism

$$\Xi({}_AM)^\# \otimes \mathbb{Q}_l \xrightarrow{{}_A\vartheta_l} \operatorname{Det}_{A_l} R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], M_l)^\#$$

is given by the composite

$$\begin{aligned}
&\operatorname{Det}_{A_l}^{-1}(\mathcal{O}_{L_m}^\times \otimes_{\mathbb{Z}} \mathbb{Q}_l) \otimes \operatorname{Det}_{A_l}(X_{\{v|\infty\}} \otimes_{\mathbb{Z}} \mathbb{Q}_l) \\
&\xrightarrow{\sim} \operatorname{Det}_{A_l}^{-1}(\mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes_{\mathbb{Z}} \mathbb{Q}_l) \otimes \operatorname{Det}_{A_l}(X_{\{v|ml\infty\}} \otimes_{\mathbb{Z}} \mathbb{Q}_l) && (5.2)\\
&\xrightarrow{\sim} \operatorname{Det}_{A_l}^{-1}(\mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes_{\mathbb{Z}} \mathbb{Q}_l) \otimes \operatorname{Det}_{A_l}(X_{\{v|ml\infty\}} \otimes_{\mathbb{Z}} \mathbb{Q}_l) && (5.3)\\
&\xrightarrow{\sim} \operatorname{Det}_{A_l} R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], M_l)^\#. && (5.4)
\end{aligned}$$

Here (5.2) is induced by the short exact sequences

$$\begin{gathered}
0 \to \mathcal{O}_{L_m}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\text{val}} Y_{\{v|ml\}} \otimes_{\mathbb{Z}} \mathbb{Q} \to 0 \\
0 \to Y_{\{v|ml\}} \otimes_{\mathbb{Z}} \mathbb{Q} \to X_{\{v|ml\infty\}} \otimes_{\mathbb{Z}} \mathbb{Q} \to X_{\{v|\infty\}} \otimes_{\mathbb{Z}} \mathbb{Q} \to 0
\end{gathered}$$

and the identity map on $Y_{\{v|ml\}}\otimes_{\mathbb{Z}}\mathbb{Q}$. The isomorphism (5.3) is multiplication with the (Euler-) factor (see [15][Lemma 2]) $\prod_{p|ml}\mathcal{E}_p^\# \in A^\times$ where $\mathcal{E}_p \in A^\times$ is defined by

$$\mathcal{E}_p = \sum_{\eta(D_p)=1} |D_p/I_p| e_\eta + \sum_{\eta(D_p)\neq 1} (1-\eta(p))^{-1} e_\eta \tag{5.5}$$

with D_p (resp. I_p) denoting the decomposition subgroup (resp. inertia subgroup) of G_m at p.

Finally (5.4) arises as follows. Put

$$\Delta(L_m) := R\operatorname{Hom}_{\mathbb{Z}_l}(R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], T_l), \mathbb{Z}_l)[-3].$$

Then $\Delta(L_m)$ is a perfect complex of $\mathfrak{A}_l$-modules and there is a natural isomorphism

$$\mathrm{Det}_{\mathfrak{A}_l}\Delta(L_m)\cong\mathrm{Det}_{\mathfrak{A}_l}R\Gamma_c(\mathbb{Z}[\frac{1}{ml}],T_l)^{\#}.$$

On the other hand, the cohomology of $\Delta(L_m)$ can be computed by Tate-Poitou duality, the Kummer sequence and some additional arguments [15][Prop. 3.3]. One finds $H^i(\Delta(L_m))=0$ for $i\neq 1,2$, a canonical isomorphism

$$H^1(\Delta(L_m))\cong H^1(\mathcal{O}_{L_m}[\frac{1}{ml}],\mathbb{Z}_l(1))\cong\mathcal{O}_{L_m}[\frac{1}{ml}]^\times\otimes_{\mathbb{Z}}\mathbb{Z}_l$$

and a short exact sequence

$$0\to\mathrm{Pic}(\mathcal{O}_{L_m}[\frac{1}{ml}])\otimes_{\mathbb{Z}}\mathbb{Z}_l\to H^2(\Delta(L_m))\to X_{\{v|ml\infty\}}\otimes_{\mathbb{Z}}\mathbb{Z}_l\to 0.$$

After tensoring with $\mathbb{Q}_l$ this computation gives the isomorphism (5.4). Conjecture 3 then becomes the statement that the element $\prod_{p|ml}\mathcal{E}_p^{\#}\cdot{}_A\vartheta_\infty(L^*({}_AM,0)^{-1})$ described in (5.1) and (5.5) is not only a basis of $\mathrm{Det}_{A_l}\Delta(L_m)\otimes\mathbb{Q}_l$ but in fact an $\mathfrak{A}_l$-basis of $\mathrm{Det}_{\mathfrak{A}_l}\Delta(L_m)$.

Iwasawa Theory. We fix l and m as in the last section and retain the notation introduced there. Put

$$\Lambda=\varprojlim_n\mathbb{Z}_l[G_{ml^n}]\cong\mathbb{Z}_l[G_{\ell m_0}][[T]]$$

where

$$m=m_0l^{\mathrm{ord}_l(m)};\quad \ell=\begin{cases}l & l\neq 2\\ 4 & l=2.\end{cases}$$

The Iwasawa algebra Λ is a finite product of complete local 2-dimensional Cohen-Macaulay (even complete intersection) rings. However, Λ is regular if and only if $l\nmid\#G_{\ell m_0}$. As usual, the element $T=\gamma-1\in\Lambda$ depends on the choice of a topological generator γ of $\mathrm{Gal}(L_{ml^\infty}/L_{\ell m_0})\cong\mathbb{Z}_l$.

Defining a perfect complex of Λ-modules

$$\Delta^\infty=\varprojlim_n\Delta(L_{m_0l^n})$$

we have $H^i(\Delta^\infty)=0$ for $i\neq 1,2$, a canonical isomorphism

$$H^1(\Delta^\infty)\cong U^\infty_{\{v|ml\}}:=\varprojlim_n\mathcal{O}_{L_{m_0l^n}}[\frac{1}{ml}]^\times\otimes_{\mathbb{Z}}\mathbb{Z}_l$$

and a short exact sequence

$$0\to P^\infty_{\{v|ml\}}\to H^2(\Delta^\infty)\to X^\infty_{\{v|ml\infty\}}\to 0$$

where

$$P^\infty_{\{v|ml\}}:=\varprojlim_n\mathrm{Pic}(\mathcal{O}_{L_{m_0l^n}}[\frac{1}{ml}])\otimes_{\mathbb{Z}}\mathbb{Z}_l,\quad X^\infty_{\{v|ml\infty\}}:=\varprojlim_n X_{\{v|m_0l\infty\}}(L_{m_0l^n})\otimes_{\mathbb{Z}}\mathbb{Z}_l.$$

All limits are taken with respect to Norm maps (on Y_S this is the map sending a place to its restriction). For $d\mid m_0$ put

$$\begin{aligned}\eta_d:=&(1-\zeta_{\ell dl^n})_{n\geq 0}\in U^\infty_{\{v|ml\}}\\ \sigma:=&(\sigma_{\ell m_0l^n})_{n\geq 0}\in Y^\infty_{\{v|ml\infty\}}\\ \theta_d:=&(g_{\ell dl^n})_{n\geq 0}\in\frac{1}{[L_{m_0}:L_d]}\cdot(\gamma-\chi_{\mathrm{cyclo}}(\gamma))^{-1}\Lambda\end{aligned}$$

where

$$g_k = -\sum_{0<a<k,(a,k)=1} \left(\frac{a}{k} - \frac{1}{2}\right) \tau_{a,k}^{-1} \in \mathbb{Q}[G_k] \tag{5.6}$$

with $\tau_{a,k} \in G_k$ defined by $\tau_{a,k}(\zeta_k) = \zeta_k^a$. Here we also view $\tau_{a,k}$ as an element of $\mathbb{Q}[G_{k'}]$ for $k \mid k'$ (which allows us to view θ_d as an element of the fraction field of Λ for $d \mid m_0$) by $\tau_{a,k} \mapsto [G_{k'} : G_k]^{-1} \sum_{a' \equiv a \mod k} \tau_{a',k'}$. The relationship between θ_d and l-adic L-functions in the usual normalization is given by the interpolation formula

$$\chi\chi_{\text{cyclo}}^j(\theta_d) = (1 - \chi^{-1}(l)l^{-j})L(\chi^{-1}, j) =: L_l(\chi^{-1}\omega^{1-j}, j) \tag{5.7}$$

for all characters χ of conductor dl^n and $j \leq 0$ (here ω denotes the Teichmueller character).

We fix an embedding $\bar{\mathbb{Q}}_l \to \mathbb{C}$ and identify $\hat{G}_k$ with the set of $\bar{\mathbb{Q}}_l$-valued characters. The total ring of fractions

$$Q(\Lambda) \cong \prod_{\psi \in \hat{G}_{\ell m_0}^{\mathbb{Q}_l}} Q(\psi) \tag{5.8}$$

of Λ is a product of fields indexed by the $\mathbb{Q}_l$-rational characters of $G_{\ell m_0}$. Since for any place w of $\mathbb{Q}$ the $\mathbb{Z}[G_{m_0 l^n}]$-module $Y_{\{v|w\}}(L_{m_0 l^n})$ is induced from the trivial module $\mathbb{Z}$ on the decomposition group $D_w \subseteq G_{m_0 l^n}$, and for $w = \infty$ (resp. nonarchimedean w) we have $[G_{m_0 l^n} : D_w] = [L_{m_0 l^n} : \mathbb{Q}]/2$ (resp. the index $[G_{m_0 l^n} : D_w]$ is bounded as $n \to \infty$) one computes easily

$$\dim_{Q(\psi)}(U^\infty_{\{v|ml\}} \otimes_\Lambda Q(\psi)) = \dim_{Q(\psi)}(Y^\infty_{\{v|ml\infty\}} \otimes_\Lambda Q(\psi)) = \begin{cases} 1 & \psi \text{ even} \\ 0 & \psi \text{ odd.} \end{cases} \tag{5.9}$$

Note that the inclusion $X^\infty_{\{v|ml\infty\}} \subseteq Y^\infty_{\{v|ml\infty\}}$ becomes an isomorphism after tensoring with $Q(\psi)$ and that $e_\psi(\eta_{m_0}^{-1} \otimes \sigma)$ is a $Q(\psi)$-basis of

$$\begin{aligned} &\mathrm{Det}^{-1}_{Q(\psi)}(U^\infty_{\{v|ml\}} \otimes_\Lambda Q(\psi)) \otimes \mathrm{Det}_{Q(\psi)}(X^\infty_{\{v|ml\infty\}} \otimes_\Lambda Q(\psi) \\ \cong &\mathrm{Det}_{Q(\psi)} (\Delta^\infty \otimes_\Lambda Q(\psi)) \end{aligned}$$

for even ψ. For odd ψ the complex $\Delta^\infty \otimes_\Lambda Q(\psi)$ is acyclic and we can view $e_\psi \theta_{m_0} \in Q(\psi)$ as an element of

$$\mathrm{Det}_{Q(\psi)} (\Delta^\infty \otimes_\Lambda Q(\psi)) \cong Q(\psi).$$

Note also that $e_\psi \theta_{m_0} = 0$ (resp. $e_\psi(\eta_{m_0}^{-1} \otimes \sigma) = 0$) if ψ is even (resp. odd). Hence we obtain an element

$$\mathcal{L} := \theta_{m_0}^{-1} + 2 \cdot \eta_{m_0}^{-1} \otimes \sigma \in \mathrm{Det}_{Q(\Lambda)} (\Delta^\infty \otimes_\Lambda Q(\Lambda)).$$

THEOREM 5.2. *There is an equality of invertible* Λ*-submodules*

$$\Lambda \cdot \mathcal{L} = \mathrm{Det}_\Lambda \Delta^\infty$$

of $\mathrm{Det}_{Q(\Lambda)} (\Delta^\infty \otimes_\Lambda Q(\Lambda))$.

This statement is an Iwasawa theoretic analogue of Conjecture 3 and, as we shall see below, it implies Conjecture 3 for $M = h^0(\operatorname{Spec} L_d)(j)$, $\mathfrak{A} = \mathbb{Z}[G_d]$ and any $d \mid m_0 l^\infty$ and $j \leq 0$. Our proof will also show that Theorem 5.2 is essentially equivalent to the main conjecture of Iwasawa theory, combined with the vanishing of μ-invariants of various Iwasawa modules. The idea to seek a conjecture that unifies the Tamagawa number conjecture on the one hand and the Iwasawa main conjecture on the other goes back to Kato [48], [49][3.3.8].

Proof of Theorem 5.2. The following Lemma allows a prime-by-prime analysis of the identity in Theorem 5.2.

Lemma 5.3. *Let R be a Noetherian Cohen-Macaulay ring with total ring of fractions $Q(R)$. Suppose R is a finite product of local rings. If I and J are invertible R-submodules of some invertible $Q(R)$-module M then $I = J$ if and only if $I_{\mathfrak{q}} = J_{\mathfrak{q}}$ (inside $M_{\mathfrak{q}}$) for all height 1 prime ideals $\mathfrak{q}$ of R.*

Proof. Since R is a product of local rings both I and J are free of rank 1 with bases $b_I, b_J \in M$, say. Since $Q(R)$ is Artinian, hence a product of local rings, M is free with basis b_M, say. Writing $b_I = \frac{x}{y} b_M$ we find that x cannot be a zero-divisor in R since I is R-free. Hence $\frac{x}{y}$ is a unit in $Q(R)$ and we may write

$$b_J = \frac{x'}{y'} b_M = \frac{x'y}{y'x} b_I =: \frac{a}{b} b_I.$$

Since R is Cohen-Macaulay and b is not a zero-divisor all prime divisors $\mathfrak{p}_1, ..., \mathfrak{p}_n$ of the principal ideal bR have height 1 [59][Thm. 17.6]. By assumption $bR_{\mathfrak{p}_i} = aR_{\mathfrak{p}_i}$ for $i = 1, .., n$. Hence $a \in \phi_i^{-1}(bR_{\mathfrak{p}_i})$ where $\phi_i : R \to R_{\mathfrak{p}_i}$ is the natural map, and the primary decomposition of the ideal bR [59][Th.6.8] gives

$$a \in \phi_1^{-1}(bR_{\mathfrak{p}_1}) \cap \cdots \cap \phi_n^{-1}(bR_{\mathfrak{p}_n}) = bR.$$

So $\frac{a}{b} \in R$ and $I = Rb_I \supseteq Rb_J = J$. By symmetry $I = J$. □

At this stage a fundamental distinction presents itself. We call a height 1 prime $\mathfrak{q}$ of Λ *regular* (resp. *singular*) if $l \notin \mathfrak{q}$ (resp. $l \in \mathfrak{q}$). For a regular prime $\mathfrak{q}$ the ring $\Lambda_{\mathfrak{q}}$ is a discrete valuation ring with fraction field $Q(\psi)$ for some $\psi = \psi_{\mathfrak{q}} \in \hat{G}_{\ell m_0}^{\mathbb{Q}_l}$. The residue field $\mathbb{Q}_l(\mathfrak{q})$ of $\Lambda_{\mathfrak{q}}$ is a finite extension of the field of values $\mathbb{Q}_l(\psi)$ of ψ. For singular primes, on the other hand, the ring $\Lambda_{\mathfrak{q}}$ is regular if and only if Λ is.

Analysis of regular primes. The computation of the cohomology of Δ^∞ given above shows that the identity

$$\Lambda_{\mathfrak{q}} \cdot \mathcal{L} = \operatorname{Det}_{\Lambda_{\mathfrak{q}}} (\Delta^\infty \otimes_\Lambda \Lambda_{\mathfrak{q}})$$

is equivalent to

$$\operatorname{Fit}_{\Lambda_{\mathfrak{q}}}(U^\infty_{\{v|ml\},\mathfrak{q}}/\Lambda_{\mathfrak{q}} \cdot \eta_{m_0}) = \operatorname{Fit}_{\Lambda_{\mathfrak{q}}}(P^\infty_{\{v|ml\},\mathfrak{q}}) \cdot \operatorname{Fit}_{\Lambda_{\mathfrak{q}}}(X^\infty_{\{v|ml\infty\},\mathfrak{q}}/\Lambda_{\mathfrak{q}} \cdot \sigma) \tag{5.10}$$

if $\psi_{\mathfrak{q}}$ is even and to

$$\theta_{m_0} \cdot \operatorname{Fit}_{\Lambda_{\mathfrak{q}}}(U^\infty_{\{v|ml\},\mathfrak{q}}) = \operatorname{Fit}_{\Lambda_{\mathfrak{q}}}(P^\infty_{\{v|ml\},\mathfrak{q}}) \cdot \operatorname{Fit}_{\Lambda_{\mathfrak{q}}}(X^\infty_{\{v|ml\infty\},\mathfrak{q}}) \tag{5.11}$$

if $\psi_{\mathfrak{q}}$ is odd. Here $\operatorname{Fit}_{\Lambda_{\mathfrak{q}}}(M)$ denotes the first Fitting (or order) ideal of any finitely generated torsion $\Lambda_{\mathfrak{q}}$-module M. Put

$$U^\infty := \varprojlim_n \mathcal{O}^\times_{L_{m_0 l^n}} \otimes_{\mathbb{Z}} \mathbb{Z}_l, \quad P^\infty := \varprojlim_n \operatorname{Pic}(\mathcal{O}_{L_{m_0 l^n}}) \otimes_{\mathbb{Z}} \mathbb{Z}_l.$$

The exact sequences of Λ-modules

$$0 \to U^\infty \to U^\infty_{\{v|ml\}} \to Y^\infty_{\{v|l\}} \to P^\infty \to P^\infty_{\{v|ml\}} \to 0 \tag{5.12}$$

and

$$0 \to X^\infty_{\{v|m_0\}} \to X^\infty_{\{v|lm\infty\}} \to Y^\infty_{\{v|l\}} \oplus Y^\infty_{\{v|\infty\}} \to 0$$

together with the fact that σ is a basis of $Y^\infty_{\{v|\infty\},\mathfrak{q}}$ for $\psi_\mathfrak{q}$ even (resp. $U^\infty_\mathfrak{q} = \mathbb{Z}_l(1)_\mathfrak{q}$ and $Y^\infty_{\{v|\infty\},\mathfrak{q}} = 0$ for $\psi_\mathfrak{q}$ odd), imply that the identities (5.10) and (5.11) are equivalent, respectively, to

$$\mathrm{Fit}_{\Lambda_\mathfrak{q}}(U^\infty_\mathfrak{q}/\Lambda_\mathfrak{q}\cdot\eta_{m_0}) = \mathrm{Fit}_{\Lambda_\mathfrak{q}}(P^\infty_\mathfrak{q})\cdot\mathrm{Fit}_{\Lambda_\mathfrak{q}}(X^\infty_{\{v|m_0\},\mathfrak{q}}) \tag{5.13}$$

and

$$\theta_{m_0}\cdot\mathrm{Fit}_{\Lambda_q}(\mathbb{Z}_l(1)_\mathfrak{q}) = \mathrm{Fit}_{\Lambda_\mathfrak{q}}(P^\infty_\mathfrak{q})\cdot\mathrm{Fit}_{\Lambda_\mathfrak{q}}(X^\infty_{\{v|m_0\},\mathfrak{q}}). \tag{5.14}$$

The following two Lemmas then finish the verification of these identities.

LEMMA 5.4. *(Main conjecture) For any regular height 1 prime $\mathfrak{q}$ of Λ let $d \mid m_0$ be such that $\psi_\mathfrak{q}$ has conductor d or $d\ell$. Put $\epsilon = 0$ or 1 according to whether $\psi_\mathfrak{q} \neq 1$ or $\psi_\mathfrak{q} = 1$. Then*

$$\mathrm{Fit}_{\Lambda_\mathfrak{q}}(U^\infty_\mathfrak{q}/\Lambda_\mathfrak{q}\cdot T^\epsilon\cdot\eta_d) = \mathrm{Fit}_{\Lambda_\mathfrak{q}}(P^\infty_\mathfrak{q})$$

if $\psi_\mathfrak{q}$ is even, and

$$\theta_d\cdot\mathrm{Fit}_{\Lambda_q}(\mathbb{Z}_l(1)_\mathfrak{q}) = \mathrm{Fit}_{\Lambda_\mathfrak{q}}(P^\infty_\mathfrak{q})$$

if $\psi_\mathfrak{q}$ is odd.

LEMMA 5.5. *For d and ϵ as in Lemma 5.4 we have*

$$\mathrm{Fit}_{\Lambda_\mathfrak{q}}(\Lambda_\mathfrak{q}\cdot T^\epsilon\cdot\eta_d/\Lambda_\mathfrak{q}\cdot\eta_{m_0}) = T^{-\epsilon}\prod_{p|m_0,\, p\nmid d}(1-\mathrm{Fr}_p^{-1}) = \mathrm{Fit}_{\Lambda_\mathfrak{q}}(X^\infty_{\{v|m_0\},\mathfrak{q}})$$

if $\psi_\mathfrak{q}$ is even and

$$\Lambda_\mathfrak{q}\cdot\theta_{m_0}/\theta_d = \prod_{p|m_0,\, p\nmid d}(1-\mathrm{Fr}_p^{-1}) = \mathrm{Fit}_{\Lambda_\mathfrak{q}}(X^\infty_{\{v|m_0\},\mathfrak{q}})$$

if $\psi_\mathfrak{q}$ is odd. Here we view the Frobenius automorphism $\mathrm{Fr}_p \in G_{m_0l^\infty}/I_p$ as an element of $G_{m_0l^\infty} \subset \Lambda$ using the fact that $I_p \cong G_{p^{\mathrm{ord}_p(m)}}$ is canonically a direct factor of $G_{m_0l^\infty}$.

PROOF OF LEMMA 5.4. Denote by C^{cyclo} the Λ-submodule of U^∞ generated by $T\eta_1$ and η_n for $1 \neq n \mid m_0$. Let ψ be an even character of $G_{\ell m_0}$. The main conjecture [41][Th. 3.1+ Rem. b) and c)] says that the characteristic ideal of the $\mathbb{Z}_l[\psi][[T]]$-module $(P^\infty)_\psi$ equals that of $(U^\infty/C^{\mathrm{cyclo}})_\psi$ where for any Λ-module M we put $M_\psi = M\otimes_{\mathbb{Z}_l[G_{\ell m_0}]}\mathbb{Z}_l[\psi] = M\otimes_\Lambda \mathbb{Z}_l[\psi][[T]]$. For any height 1 prime ideal $\mathfrak{q}$ with $\psi_\mathfrak{q} = \psi$ the map $\Lambda \to \Lambda_\mathfrak{q}$ factors through $\Lambda \to \mathbb{Z}_l[\psi][[T]]$ and we deduce $\mathrm{Fit}_{\Lambda_\mathfrak{q}}(U^\infty_\mathfrak{q}/C^{\mathrm{cyclo}}_\mathfrak{q}) = \mathrm{Fit}_{\Lambda_\mathfrak{q}}(P^\infty_\mathfrak{q})$. It remains to show that $C^{\mathrm{cyclo}}_\mathfrak{q}$ is generated by η_d over $\Lambda_\mathfrak{q}$ (or $T\eta_1$ if $\psi_\mathfrak{q} = 1$). This follows from the distribution (or Euler system) relations satisfied by the η_n. For $d \mid m_0$ put

$$N_d = \left(\sum_{\tau\in\mathrm{Gal}(L_{m_0}/L_d)}\tau\right) \in \Lambda.$$

Then for $d \mid n \mid m_0$

$$N_d \cdot \eta_n = [L_{m_0} : L_n]\mathrm{Norm}_{L_n/L_d}\eta_n = [L_{m_0} : L_n]\left(\prod_{p|n,\, p\nmid d}(1-\mathrm{Fr}_p^{-1})\right)\eta_d. \tag{5.15}$$

Since $\psi(N_d) = [L_{m_0} : L_d] \neq 0$ the element N_d is a unit in $\Lambda_{\mathfrak{q}}$ and we have therefore expressed η_n as a $\Lambda_{\mathfrak{q}}$-multiple of η_d. For $d \nmid n$ on the other hand, ψ is a nontrivial character of $\mathrm{Gal}(L_{m_0}/L_n)$ and we have

$$(\eta_n)_{\mathfrak{q}} = [L_{m_0} : L_n]^{-1}(N_n\eta_n)_{\mathfrak{q}} = [L_{m_0} : L_n]^{-1}\psi(N_n)\eta_{n,\mathfrak{q}} = 0.$$

Let now ψ be an odd character of $G_{\ell m_0}$ and ω the Teichmueller character. The main conjecture [41][Th. 3.2] then asserts that the characteristic ideal of the $\mathbb{Z}_l[\psi][[T]]$-module $(P^\infty)_\psi$ equals $e_\psi\frac{1}{2}\theta_d$ if $\psi \neq \omega$. For $\psi = \omega$ the statement of [41][Th. 3.2] is that both

$$\frac{1}{2}\theta_1 \cdot \mathrm{Fit}_{\mathbb{Z}_l[[T]]}(\mathbb{Z}_l(1)) = \frac{1}{2}\theta_1 \cdot (\gamma - \chi_{\mathrm{cyclo}}(\gamma)) \cdot \mathbb{Z}_l[[T]]$$

and the characteristic ideal of $(P^\infty)_\omega$ are equal to $\mathbb{Z}_l[[T]]$.

We note that the main conjecture for an even character ψ is equivalent to the main conjecture for the odd character $\omega\psi^{-1}$ by an argument involving duality and Kummer theory (see the proof of Th. 3.2 in [41]). □

Proof of Lemma 5.5. For any prime $p \mid m_0$ the decomposition group $D_p \subseteq G_{m_0 l^\infty}$ has finite index and the inertia subgroup $I_p \subseteq D_p$ is finite. Moreover one has a direct product decomposition $D_p = I_p \times \overline{< \mathrm{Fr}_p >}$. We have an isomorphism of Λ-modules

$$Y^\infty_{\{v|p\}} \cong \mathrm{Ind}_{D_p}^{G_{m_0 l^\infty}} \mathbb{Z}_l$$

and an isomorphism of $\mathbb{Z}_l[[D_p]]$-modules

$$\mathbb{Z}_l \cong \mathbb{Z}_l[[D_p]]/ < g-1 | g \in I_p; 1 - \mathrm{Fr}_p^{-1} > .$$

So if $\psi_{\mathfrak{q}}|_{I_p} \neq 1$, i.e. $p \mid d$, then $Y^\infty_{\{v|p\},\mathfrak{q}} = 0$ and if $\psi_{\mathfrak{q}}|_{I_p} = 1$ the characteristic ideal of $Y^\infty_{\{v|p\},\mathfrak{q}}$ is generated by $1 - \mathrm{Fr}_p^{-1}$. The exact sequence of Λ-modules

$$0 \to X^\infty_{\{v|m_0\}} \to Y^\infty_{\{v|m_0\}} \to \mathbb{Z}_l \to 0$$

accounts for the term T^ϵ. This verifies the second equalities in the two displayed equations of Lemma 5.5.

The respective first equalities follow for even $\psi_{\mathfrak{q}}$ from the Euler system relations (5.15) with $n = m_0$ together with the fact that N_d is a unit in $\Lambda_{\mathfrak{q}}$. For odd $\psi_{\mathfrak{q}}$ we also have an Euler system relation

$$N_d \cdot \theta_{m_0} = [L_{m_0} : L_d] \cdot \left(\prod_{p|m_0,\, p\nmid d}(1-\mathrm{Fr}_p^{-1})\right)\theta_d \tag{5.16}$$

which expresses θ_{m_0} as a $\Lambda_{\mathfrak{q}}$-multiple of θ_d since N_d is a unit in $\Lambda_{\mathfrak{q}}$ (note here our convention that we view the element $\theta_d \in \mathbb{Z}_l[[G_{dl^\infty}]]$ as the element $[L_{m_0} : L_d]^{-1}N_d \cdot \tilde{\theta}_d \in \Lambda$ where $\tilde{}$ denotes any lift). □

Analysis of singular primes. The singular primes of Λ are in bijection with the $\mathbb{Q}_l$-rational characters of $G_{\ell m_0}$ of order prime to l. For a singular height 1 prime $\mathfrak{q}$ we denote by $\psi_{\mathfrak{q}}$ the corresponding character with $\mathbb{Q}_l$-rational idempotent $e_{\psi_{\mathfrak{q}}} \in \Lambda$.

The following Lemma relates μ-invariants to localization at singular primes.

LEMMA 5.6. *Let M be a finitely generated torsion Λ-module, $\mathfrak{q}$ a singular prime with character $\psi = \psi_{\mathfrak{q}}$ and χ a $\bar{\mathbb{Q}}_l$-valued character of $G_{\ell m_0}$ with prime to l-part ψ. The following are equivalent*

(i) *The μ-invariant of the $\mathbb{Z}_l[[T]]$-module $e_\psi M$ vanishes.*

(ii) *The μ-invariant of the $\mathbb{Z}_l[\chi][[T]]$-module*

$$M_\chi := M \otimes_{\mathbb{Z}[G_{m_0\ell}]} \mathbb{Z}_l[\chi] \cong e_\psi M \otimes_{\mathbb{Z}[G_{m_0\ell}]} \mathbb{Z}_l[\chi]$$

vanishes.

(iii) $M_{\mathfrak{q}} = 0$.

PROOF. It is well known that the μ-invariant in (i) (resp.(ii)) vanishes if and only if $(e_\psi M)_{(l)} = 0$ (resp. $M_{\chi,(\pi)} = 0$) where π is a uniformizer of $\mathbb{Z}_l[\chi]$. Since $\mathfrak{q}$ is the radical of (l) in $\Lambda_{\mathfrak{q}}$ the map $(e_\psi M)_{(l)} \to (e_\psi M)_{\mathfrak{q}} = M_{\mathfrak{q}}$ is an isomorphism which gives the equivalence of (i) and (iii). The map $M/\mathfrak{q}M \to M_\chi/\pi M_\chi$ is an isomorphism which gives the equivalence of (ii) and (iii) by Nakayama's Lemma. □

Unlike the case of regular primes, we not only need the equality of the μ-invariant of an Iwasawa module and a p-adic L-function but the vanishing of both.

We now analyze the localization of Δ^∞ at a singular prime $\mathfrak{q}$. By the theorem of Ferrero and Washington [75][Thm. 7.15] the μ-invariant of P^∞ vanishes. The module $X^\infty_{\{v|ml\}}$ is finite free over $\mathbb{Z}_l$ and hence has vanishing μ-invariant. The surjection $P^\infty \to P^\infty_{\{v|ml\}}$ and the exact sequence

$$0 \to X^\infty_{\{v|ml\}} \to X^\infty_{\{v|ml\infty\}} \to Y^\infty_{\{v|\infty\}} \to 0$$

then show that

$$H^2(\Delta^\infty)_{\mathfrak{q}} = Y^\infty_{\{v|\infty\},\mathfrak{q}} \cong (\Lambda_{\mathfrak{q}}/(c-1)) \cdot \sigma$$

where $c \in G_{\ell m_0}$ is the complex conjugation. Concerning $H^1(\Delta^\infty)_{\mathfrak{q}}$ one shows that η_{m_0} is a generator using the fact that all graded pieces of the filtration

$$\Lambda \cdot \eta_{m_0} \subseteq C^{\mathrm{cyclo}} \subseteq U^\infty \subseteq U^\infty_{\{v|ml\}}$$

have vanishing μ-invariant. The quotient $U^\infty_{\{v|ml\}}/U^\infty$ injects into the finite free $\mathbb{Z}_l$-module $Y^\infty_{\{v|l\}}$ and hence has vanishing μ-invariant. The quotient $U^\infty/C^{\mathrm{cyclo}}$ has vanishing μ-invariant by the main result of C. Greither's appendix to this article. Finally, the quotient $C^{\mathrm{cyclo}}_{\mathfrak{q}}/\Lambda_{\mathfrak{q}} \cdot \eta_{m_0}$ is zero by (5.15) and the fact that $1 - \mathrm{Fr}_p^{-1} \in \Lambda_{\mathfrak{q}}^\times$ for $p \mid m_0$. This follows because Fr_p has infinite order in $G_{m_0 l^\infty}$: We have $\mathrm{Fr}_p^{-N} = (1+T)^\nu$ for some integer N and $0 \neq \nu \in \mathbb{Z}_l$. So the image $1-(1+T)^\nu$ of $1 - \mathrm{Fr}_p^{-N}$ in $\Lambda/\mathfrak{q} \cong \mathbb{F}_l(\psi_{\mathfrak{q}})[[T]]$ is nonzero, and hence so is the image of $1 - \mathrm{Fr}_p^{-1}$.

Having established that η_{m_0} is a generator of $H^1(\Delta^\infty)_{\mathfrak{q}}$ one notes that

$$H^1(\Delta^\infty)_{\mathfrak{q}} \cong (\Lambda_{\mathfrak{q}}/(c-1)) \cdot \eta_{m_0}$$

since the image of η_{m_0} in the scalar extension to $Q(\psi)$ vanishes precisely for odd ψ. So for $l \neq 2$ we conclude that $H^1(\Delta^\infty)_{\mathfrak{q}}$ (resp. $H^2(\Delta^\infty)_{\mathfrak{q}}$) is free with basis η_{m_0} (resp. σ) if $\psi_{\mathfrak{q}}$ is even, whereas $H^i(\Delta^\infty)_{\mathfrak{q}} = 0$ for $i = 1, 2$ if $\psi_{\mathfrak{q}}$ is odd. This proves the $\mathfrak{q}$-part of Theorem 5.2 for $\psi_{\mathfrak{q}}$ even, and for $\psi_{\mathfrak{q}}$ odd it remains to remark that θ_{m_0} is a unit in $\Lambda_{\mathfrak{q}}$ (otherwise there would be a $\bar{\mathbb{Q}}_l$-character χ of $G_{m_0\ell}$ so that all coefficients of $\chi(\theta_{m_0}) \in \mathbb{Z}_l[\chi][[T]]$ have positive l-adic valuation which contradicts [75][p.131]) .

For $l = 2$ we conclude that the cohomology modules $H^i(\Delta^\infty)_\mathfrak{q}$ are not of finite projective dimension over $\Lambda_\mathfrak{q}$ and hence that we cannot pass to cohomology when computing $\mathrm{Det}_{\Lambda_\mathfrak{q}}(\Delta^\infty_\mathfrak{q})$. However, the complex $\Delta^\infty_\mathfrak{q}$ represents a class in

$$\mathrm{Ext}^2_{\Lambda_\mathfrak{q}}(H^2(\Delta^\infty)_\mathfrak{q}, H^1(\Delta^\infty)_\mathfrak{q}) \cong \mathrm{Ext}^2_{\Lambda_\mathfrak{q}}(\Lambda_\mathfrak{q}/(c-1), \Lambda_\mathfrak{q}/(c-1))$$

and we have $\mathrm{Ext}^i_{\Lambda_\mathfrak{q}}(\Lambda_\mathfrak{q}/(c-1), \Lambda_\mathfrak{q}/(c-1)) \cong \Lambda_\mathfrak{q}/(2, c-1)$ for even $i \geq 2$ as can be seen from the projective resolution

$$\cdots \to \Lambda_\mathfrak{q} \xrightarrow{1-c} \Lambda_\mathfrak{q} \xrightarrow{1+c} \Lambda_\mathfrak{q} \xrightarrow{1-c} \Lambda_\mathfrak{q} \to \Lambda_\mathfrak{q}/(c-1) \to 0.$$

The complex $\Delta^\infty_\mathfrak{q}$ can be constructed as the pushout

$$\begin{array}{ccccccccc} 0 & \to & (1+c)\cdot\Lambda_\mathfrak{q} & \to & \Lambda_\mathfrak{q} & \xrightarrow{1-c} & \Lambda_\mathfrak{q} & \to \Lambda_\mathfrak{q}/(c-1) & \to 0 \\ & & \mu\downarrow & & \downarrow & & \| & \| & \\ 0 & \to & \Lambda_\mathfrak{q}/(c-1) & \to & \Delta^{\infty,1}_\mathfrak{q} & \to & \Delta^{\infty,2}_\mathfrak{q} & \to \Lambda_\mathfrak{q}/(c-1) & \to 0 \end{array} \tag{5.17}$$

via some homomorphism μ, determined by $\mu(1+c)$. This diagram induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^i_{\Lambda_\mathfrak{q}}((c+1)\cdot\Lambda_\mathfrak{q}, \Lambda_\mathfrak{q}/(c-1)) & \longrightarrow & \mathrm{Ext}^{i+2}_{\Lambda_\mathfrak{q}}(\Lambda_\mathfrak{q}/(c-1), \Lambda_\mathfrak{q}/(c-1)) \\ \mu_*\downarrow & & \| \\ \mathrm{Ext}^i_{\Lambda_\mathfrak{q}}(\Lambda_\mathfrak{q}/(c-1), \Lambda_\mathfrak{q}/(c-1)) & \longrightarrow & \mathrm{Ext}^{i+2}_{\Lambda_\mathfrak{q}}(\Lambda_\mathfrak{q}/(c-1), \Lambda_\mathfrak{q}/(c-1)) \end{array}$$

where the horizontal maps are isomorphisms for large i since $\Delta^{\infty,1}_\mathfrak{q}$ and $\Delta^{\infty,2}_\mathfrak{q}$ have finite projective dimension (as $\Delta^\infty_\mathfrak{q}$ is perfect). It follows that μ_* is an isomorphism. Since both the source and target of μ_* are isomorphic to $\Lambda_\mathfrak{q}/(2, c-1)$ for i even, μ_* is given by multiplication with a unit in $\Lambda_\mathfrak{q}$ (this ring being local) and hence $\mu(1+c) \in \Lambda_\mathfrak{q}/(c-1)^\times$. This means μ is an isomorphism and we conclude that $\Delta^\infty_\mathfrak{q}$ is quasi-isomorphic to the top row in (5.17). We may pick bases γ_i of $\Delta^{\infty,i}_\mathfrak{q}$ so that $\gamma_2 \mapsto \sigma$ and $(c+1)\gamma_1 = \eta_{m_0}$. It remains to verify that $\gamma_1^{-1} \otimes \gamma_2 = u \cdot \mathcal{L}$ for some $u \in \Lambda_\mathfrak{q}^\times$, and this we may check in $Q(\psi)$ for all $\psi \in \hat{G}^{\mathbb{Q}_l}_{\ell m_0}$ which induce $\psi_\mathfrak{q}$. For ψ even we have in $Q(\psi)$

$$\gamma_1^{-1} \otimes \gamma_2 = (\frac{1}{2} \cdot \eta_{m_0})^{-1} \otimes \sigma = \mathcal{L}$$

and for ψ odd we note that $\gamma_1^{-1} \otimes \ 2 \cdot \gamma_2$ is the canonical basis of $\Delta^\infty \otimes_\Lambda Q(\psi)$ arising from the fact that this complex is acyclic. We then have

$$\gamma_1^{-1} \otimes \gamma_2 = \frac{1}{2} \cdot \gamma_1^{-1} \otimes \ 2 \cdot \gamma_2 = u \cdot \theta_{m_0}^{-1} = u \cdot \mathcal{L}$$

where $u := \frac{\theta_{m_0}}{2}$ is a unit in $\Lambda_\mathfrak{q}$ by the remark in [75][p.131] already used in the case $l \neq 2$. $\square$

The descent argument for $j = 0$. We now indicate how Theorem 5.2 implies the l-part of Conjecture 3 for $M = h^0(\mathrm{Spec}(L_m))$ and $\mathfrak{A} = \mathbb{Z}[G_m]$. We have a ring homomorphism

$$\Lambda \to \mathbb{Z}_l[G_m] = \mathfrak{A}_l \subseteq A_l = \prod_{\chi \in \hat{G}_m^{\mathbb{Q}_l}} \mathbb{Q}_l(\chi), \tag{5.18}$$

a canonical isomorphism of perfect complexes

$$\Delta^\infty \otimes^{\mathbb{L}}_\Lambda \mathfrak{A}_l \cong \Delta(L_m)$$

and a canonical isomorphism of determinants

$$(\mathrm{Det}_\Lambda \Delta^\infty) \otimes_\Lambda \mathfrak{A}_l \cong \mathrm{Det}_{\mathfrak{A}_l} \Delta(L_m).$$

Given Theorem 5.2, the element $\mathcal{L} \otimes 1$ is an $\mathfrak{A}_l$-basis of $\mathrm{Det}_{\mathfrak{A}_l} \Delta(L_m)$ under this isomorphism. It remains then to verify that the image of $\mathcal{L} \otimes 1$ in $\mathrm{Det}_{A_l}(\Delta(L_m) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$ agrees with that of ${}_A\vartheta_\infty(L^*({}_AM, 0)^{-1})$. Luckily, this is again a computation taking place over the algebra A_l which avoids any delicate analysis over the possibly non-regular ring $\mathfrak{A}_l$, and which can be performed character by character. Denote by

$$\begin{aligned} \phi : \mathrm{Det}_{\mathbb{Q}_l(\chi)}&(\Delta(L_m) \otimes_{\mathfrak{A}_l} \mathbb{Q}_l(\chi)) \\ &\cong \begin{cases} \mathrm{Det}^{-1}_{\mathbb{Q}_l(\chi)}(\mathcal{O}^\times_{L_m} \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)) \otimes \mathrm{Det}_{\mathbb{Q}_l(\chi)}(X_{\{v|\infty\}} \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)) & \chi \neq 1 \text{ even} \\ \mathbb{Q}_l(\chi) & \text{otherwise} \end{cases} \end{aligned} \tag{5.19}$$

the isomorphism induced by (5.2). Then in view of (5.1) and (5.3), for each $\chi \in \hat{G}_m^{\mathbb{Q}_l}$ we must show

$$\prod_{p|ml} (\mathcal{E}_p^\#)^{-1} \phi(\mathcal{L} \otimes 1) = \begin{cases} 2[L_m : L_{f_\chi}][1 - \zeta_{f_\chi}]^{-1} \otimes \sigma_m & \chi \neq 1 \text{ even} \\ \left(L(\chi, 0)^\#\right)^{-1} & \text{otherwise.} \end{cases} \tag{5.20}$$

In the remainder of this section we verify the identity (5.20). We also denote by χ the composite ring homomorphism $\Lambda \to \mathbb{Q}_l(\chi)$ in (5.18), and by $\mathfrak{q}_\chi$ its kernel. Then $\mathfrak{q}_\chi$ is a regular prime of Λ and $\Lambda_{\mathfrak{q}_\chi}$ is a discrete valuation ring with residue field $\mathbb{Q}_l(\chi)$ and fraction field some direct factor of $Q(\Lambda)$ (indexed in (5.8) by the character ψ of $G_{\ell m_0}$ obtained by the unique decomposition $\chi = \psi \times \eta$ where η is a character of $\mathrm{Gal}(L_{ml^\infty}/L_{\ell m_0})$). We may view $\mathcal{L}$ as a $\Lambda_{\mathfrak{q}_\chi}$-basis of $(\mathrm{Det}_\Lambda \Delta^\infty)_{\mathfrak{q}_\chi}$. The following Lemma is the key ingredient in the descent computation.

Lemma 5.7. *Let R be a discrete valuation ring with fraction field F, residue field k and uniformiser ϖ. Suppose Δ is a perfect complex of R-modules so that the R-torsion subgroup of each $H^i(\Delta)$ is annihilated by ϖ. Define free R-modules M^i by the short exact sequence*

$$0 \to H^i(\Delta)_\varpi \to H^i(\Delta) \to M^i \to 0. \tag{5.21}$$

Together with the exact sequences of k-vector spaces

$$0 \to H^i(\Delta)/\varpi \to H^i(\Delta \otimes^{\mathbb{L}}_R k) \to H^{i+1}(\Delta)_\varpi \to 0 \tag{5.22}$$

induced by the exact triangle in the derived category of R-modules

$$\Delta \xrightarrow{\varpi} \Delta \to \Delta \otimes^{\mathbb{L}}_R k \to \Delta[1]$$

we find an isomorphism

$$\begin{aligned}\mathrm{Det}_k H^i(\Delta\otimes^{\mathbb{L}}_R k) &\cong \mathrm{Det}_k(H^i(\Delta)/\varpi)\otimes_k \mathrm{Det}_k(H^{i+1}(\Delta)_\varpi)\\ &\cong \mathrm{Det}_k(H^i(\Delta)_\varpi)\otimes_k \mathrm{Det}_k(M^i/\varpi)\otimes_k \mathrm{Det}_k(H^{i+1}(\Delta)_\varpi)\end{aligned}$$

and hence an isomorphism

$$\phi_\varpi : \mathrm{Det}_k\left(\Delta\otimes^{\mathbb{L}}_R k\right)\cong \bigotimes_{i\in\mathbb{Z}}\mathrm{Det}_k(M^i/\varpi)^{(-1)^i}.$$

For each i fix an R-basis β_i of $\mathrm{Det}_R(M^i)$. Let $e\in\mathbb{Z}$ be such that $b_\varpi=\varpi^e\bigotimes_{i\in\mathbb{Z}}(\beta_i)^{(-1)^i}$ is an R-basis of

$$\mathrm{Det}_R\,\Delta\subseteq \mathrm{Det}_F(\Delta\otimes_R F)\cong\bigotimes_{i\in\mathbb{Z}}\left(\mathrm{Det}_F(M^i\otimes_R F)\right)^{(-1)^i}.$$

Then the image of $b_\varpi\otimes 1$ under the isomorphism

$$(\mathrm{Det}_R\,\Delta)\otimes_R k\cong \mathrm{Det}_k(\Delta\otimes^{\mathbb{L}}_R k)\xrightarrow{\phi_\varpi}\bigotimes_{i\in\mathbb{Z}}\mathrm{Det}_k(M^i/\varpi)^{(-1)^i}$$

is given by $\bigotimes_{i\in\mathbb{Z}}(\bar\beta_i)^{(-1)^i}$.

Remark. A change of uniformizer ϖ will change b_ϖ as well as the isomorphism ϕ_ϖ (unless $e=0$, e.g. if each $H^i(\Delta)$ is R-free). If $M^i=0$ for all i then $\det_R(M^i)\cong R$ canonically and we may take $\beta_i=1$. In this case the Lemma recovers the statement of [18] [Lemma 8.1] where the condition that $H^i(\Delta)$ is annihilated by ϖ is called "semisimplicity at zero" (the motivating example being a $\mathbb{Z}_l[[T]]$-module whose localisation at the prime (T) is semisimple).

Proof. Suppose $\Delta^\bullet$ is a bounded complex of finitely generated free R-modules quasi-isomorphic to Δ. Let $\lambda_i^{(k)},\mu_i^{(l)}\in\Delta^i$ (where k and l run through two index sets depending on i) be elements whose images under δ^i form an R-basis of $\mathrm{im}(\delta^i)$ and so that $\delta^i(\mu_i^{(l)})$ is an R-basis of $\mathrm{im}(\delta^i)\cap\varpi\ker(\delta^{i+1})=\mathrm{im}(\delta^i)\cap\varpi\Delta^{i+1}$. Let $\beta_i^{(n)}\in\Delta^i$ map to an R-basis of M^i. Then

$$\delta^{i-1}(\lambda_{i-1}^{(k)}),\frac{\delta^{i-1}(\mu_{i-1}^{(l)})}{\varpi},\beta_i^{(n)}$$

is an R-basis of $\ker(\delta^i)$ (the cardinality of the set $\{\mu_{i-1}^{(l)}\}$ is just $\dim_k H^i(\Delta)_\varpi$) and

$$\delta^{i-1}(\lambda_{i-1}^{(k)}),\frac{\delta^{i-1}(\mu_{i-1}^{(l)})}{\varpi},\beta_i^{(n)},\mu_i^{(l)},\lambda_i^{(k)}\tag{5.23}$$

is an R-basis of Δ^i. Set $\mu_i=\bigwedge_l\mu_i^{(l)}$, $\lambda_i=\bigwedge_k\lambda_i^{(k)}$, $\beta_i=\bigwedge_n\beta_i^{(n)}$ and define $\gamma_i\in\mathrm{Det}_F(\Delta^i\otimes_R F)^{(-1)^i}$ by

$$\gamma_i=\varpi^{(-1)^{i+1}\dim_k(H^i(\Delta)_\varpi)}\delta(\lambda_{i-1})^{(-1)^i}\wedge\delta(\mu_{i-1})^{(-1)^i}\wedge\beta_i^{(-1)^i}\wedge\mu_i^{(-1)^i}\wedge\lambda_i^{(-1)^i}.$$

Then γ_i is an R-basis of $(\mathrm{Det}_R\,\Delta^i)^{(-1)^i}$ and

$$b=\bigotimes_{i\in\mathbb{Z}}\gamma_i\mapsto\varpi^e\bigotimes_{i\in\mathbb{Z}}\beta_i^{(-1)^i};\quad e=\sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_k(H^i(\Delta)_\varpi)$$

under the isomorphism $\mathrm{Det}_F(\Delta^\bullet \otimes_R F) \cong \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_F H^i(\Delta^\bullet \otimes_R F)^{(-1)^i}$. Now $\overline{\delta(\lambda_{i-1}^{(k)})}, \overline{\frac{\delta(\mu_{i-1}^{(l)})}{\varpi}}, \overline{b_i^{(n)}}, \overline{\mu_i^{(l)}}, \overline{\lambda_i^{(k)}}$ is a k-basis of $\overline{\Delta}^i := \Delta^i \otimes_R k = \Delta^i/\varpi$,

$$\overline{\delta(\lambda_{i-1}^{(k)})}, \overline{\frac{\delta(\mu_{i-1}^{(l)})}{\varpi}}, \overline{\beta_i^{(n)}}, \overline{\mu_i^{(l)}}$$

is a k-basis of $\ker(\overline{\delta}^i)$ and the images of $\overline{\frac{\delta(\mu_{i-1}^{(l)})}{\varpi}}, \overline{\beta_i^{(n)}}, \overline{\mu_i^{(l)}}$ are a k-basis of $H^i(\Delta^\bullet/\varpi)$. The isomorphism

$$(\mathrm{Det}_R\,\Delta^\bullet) \otimes_R k = \mathrm{Det}_k(\Delta^\bullet \otimes_R k) \cong \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_k\, H^i(\Delta^\bullet \otimes_R k)^{(-1)^i}$$

sends $b \otimes 1 = \bar{b} = \bigotimes_{i\in\mathbb{Z}} \bar{\gamma}_i$ to

$$\bigotimes_{i\in\mathbb{Z}} \left(\bigwedge_l \overline{\frac{\delta(\mu_{i-1}^{(l)})}{\varpi}} \right)^{(-1)^i} \wedge \overline{\beta}_i^{(-1)^i} \wedge \overline{\mu}_i^{(-1)^i}.$$

The third map in (5.22) arises as the connecting homomorphism in the short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta^i & \xrightarrow{\varpi} & \Delta^i & \longrightarrow & \Delta^i/\varpi & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta^i} & & \downarrow{\scriptstyle \delta^i} & & \downarrow{\scriptstyle \overline{\delta}^i} & & \\
0 & \longrightarrow & \Delta^{i+1} & \xrightarrow{\varpi} & \Delta^{i+1} & \longrightarrow & \Delta^{i+1}/\varpi & \longrightarrow & 0
\end{array}$$

and therefore sends $\overline{\mu_i^{(l)}}$ to $\frac{\delta^i(\mu_i^{(l)})}{\varpi} \in H^{i+1}(\Delta)_\varpi$ and to $\overline{\frac{\delta^i(\mu_i^{(l)})}{\varpi}}$ in $H^{i+1}(\Delta)/\varpi$. The construction of ϕ_ϖ via the sequences (5.22) and (5.21) then shows that $\phi_\varpi(\overline{b}) = \bigotimes_{i\in\mathbb{Z}} \overline{\beta}_i^{(-1)^i}$. We remark that the ordering of the terms in γ_i is adapted to the particular short exact sequences inducing the isomorphism between the determinant of a complex and that of its cohomology, as well as to the sequences (5.22) and (5.21) involved in ϕ_ϖ. A different ordering would induce signs in various steps of the above computation which of course would cancel out eventually. □

For any $\mathbb{Q}_l$-rational character χ of G_m we apply this Lemma to

$$R = \Lambda_{\mathfrak{q}_\chi}, \quad \Delta = \Delta^\infty_{\mathfrak{q}_\chi}, \quad \varpi = 1 - \gamma^{l^n}$$

where γ^{l^n} is a topological generator of $\mathrm{Gal}(L_{ml^\infty}/L_{m_1})$ with $m_1 = m$ if $l \mid m$ and $m_1 = \ell m$ if $l \nmid m$. Then the cohomology of $\Delta \otimes^{\mathbb{L}}_R k = \Delta(L_m) \otimes_{\mathfrak{A}_l} \mathbb{Q}_l(\chi)$ is concentrated in degrees 1 and 2 and ϕ_ϖ is induced by the exact sequence of $k = \mathbb{Q}_l(\chi)$-vector spaces

$$0 \to M^1/\varpi \to H^1(\Delta \otimes^{\mathbb{L}}_R k) \xrightarrow{\beta_\varpi} H^2(\Delta \otimes^{\mathbb{L}}_R k) \to M^2/\varpi \to 0 \tag{5.24}$$

where β_ϖ (a so called Bockstein map) is the composite

$$H^1(\Delta \otimes^{\mathbb{L}}_R k) \to H^2(\Delta)_\varpi \to H^2(\Delta)/\varpi \to H^2(\Delta \otimes^{\mathbb{L}}_R k).$$

Note that ϖ depends only on L_m and not on χ; indeed one can construct a Bockstein map before localizing at $\mathfrak{q}_\chi$, described in the following Lemma.

Lemma 5.8. *Define for* $p \mid m_0$ *the element* $c_p \in \mathbb{Z}_l$ *by* $\gamma^{c_p l^n} = \mathrm{Fr}_p^{-f_p}$ *where* $f_p \in \mathbb{Z}$ *is the inertial degree at* p *of* $L_m/\mathbb{Q}$ *(this is an identity in* Λ *modulo multiplication with* $\mathrm{Gal}(L_{m_1}/L_m)$*). Put* $c_l = \log_l(\chi_{\mathrm{cyclo}}(\gamma^{l^n}))^{-1} \in \mathbb{Q}_l$*. Then* β_ϖ *is induced by the map*

$$H^1(\Delta(L_m)) \otimes \mathbb{Q}_l = \mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes \mathbb{Q}_l \to X_{\{v|ml\infty\}} \otimes \mathbb{Q}_l = H^2(\Delta(L_m)) \otimes \mathbb{Q}_l$$

given by

$$u \mapsto \sum_{p|m_0} c_p \sum_{v|p} \mathrm{ord}_v(u) \cdot v + c_l \sum_{v|l} \mathrm{Tr}_{L_{m,v}/\mathbb{Q}_l}(\log_l(u_v)) \cdot v.$$

Remark. One can verify directly that this last map has image in $X_{\{v|ml\infty\}} \otimes \mathbb{Q}_l$ and not only in $Y_{\{v|ml\infty\}} \otimes \mathbb{Q}_l$ (although this is of course also a consequence of Lemma 5.8). Denoting by $Nu \in \mathbb{Z}[\frac{1}{ml}]^\times$ the norm of u we have $\sum_{v|p} \mathrm{ord}_v(u) = f_p^{-1} \mathrm{ord}_p(Nu)$ and $\sum_{v|l} \mathrm{Tr}_{L_{m,v}/\mathbb{Q}_l}(\log_l(u_v)) = \log_l(Nu)$. The required identity $\sum_{p|m_0} c_p f_p^{-1} \mathrm{ord}_p(x) + c_l \log_l(x) = 0$ holds for $x = -1$, for $x = l$ (since $\log_l(l) = 0$) and for $x = p$ (applying $\log_l \chi_{\mathrm{cyclo}}$ to the defining identity of c_p we find $c_p c_l^{-1} = c_p \log_l \chi_{\mathrm{cyclo}}(\gamma^{l^n}) = -f_p \log_l(p)$), hence for all $x \in \mathbb{Z}[\frac{1}{ml}]^\times$.

Proof. The computation of the cohomology of $\Delta(L_m)$ given above arises from an exact triangle

$$\tau^{\leq 2} R\Gamma(\mathbb{Z}[\frac{1}{ml}], T_l^*(1)) \to \Delta(L_m) \to Y_{\{v|\infty\}} \otimes_{\mathbb{Z}} \mathbb{Z}_l[-2]$$

(where the truncation $\tau^{\leq 2}$ is only necessary for $l = 2$). Passing to the inverse limit we find that there is an exact triangle of Λ-modules

$$\tau^{\leq 2} R\Gamma(\mathbb{Z}[\frac{1}{ml}], T_l^*(1)^\infty) \to \Delta^\infty \to Y^\infty_{\{v|\infty\}}[-2]$$

which induces, after localisation at $\mathfrak{q}_\chi$, a commutative diagram of Bockstein maps

$$\begin{array}{ccccc} H^1(\mathcal{O}_{L_m}[\frac{1}{ml}], \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l(\chi) & \to & H^1(\Delta(L_m)) \otimes \mathbb{Q}_l(\chi) & \to & 0 \\ \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta_\varpi} & & \downarrow \\ H^2(\mathcal{O}_{L_m}[\frac{1}{ml}], \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l(\chi) & \to & H^2(\Delta(L_m)) \otimes \mathbb{Q}_l(\chi) & \to & Y_{\{v|\infty\}} \otimes \mathbb{Q}_l(\chi). \end{array}$$

This shows that the image of β_ϖ, has no components at the infinite places. In order to compute β' we apply the following Lemma to $G = \pi_1^{et}(\mathrm{Spec}(\mathcal{O}_{L_{m_1}}[\frac{1}{ml}]))$, $\Gamma = \mathrm{Gal}(L_{ml^\infty}/L_{m_1})$, $\gamma_0 = \gamma^{l^n}$ and $M = \mathbb{Z}_l(1)$.

Lemma 5.9. *Let* Γ *be a free* $\mathbb{Z}_l$*-module of rank 1 with generator* γ_0 *and* $G \to \Gamma$ *a surjection of profinite groups. Denote by* $\theta \in H^1(G, \mathbb{Z}_l) = \mathrm{Hom}(G, \mathbb{Z}_l)$ *the unique homomorphism factoring through* Γ *with* $\theta(\gamma_0) = 1$ *and put* $\Lambda = \mathbb{Z}_l[[\Gamma]]$*. For any continuous* $\mathbb{Z}_l[[G]]$*-module* M *we have an exact triangle in the derived category of* Λ*-modules*

$$R\Gamma(G, M \otimes \Lambda) \xrightarrow{1-\gamma_0} R\Gamma(G, M \otimes \Lambda) \to R\Gamma(G, M \otimes \Lambda) \otimes^{\mathbb{L}}_\Lambda \mathbb{Z}_l \cong R\Gamma(G, M).$$

Then the Bockstein map

$$\beta^i : H^i(G, M) \to H^{i+1}(G, M)$$

arising from this triangle coincides with the cup product $\theta \cup -$*.*

PROOF. See [63][Lemma 1.2] . □

This Lemma describes β_ϖ as being induced by cup product with θ over the field L_{m_1}. Now in case that $m_1 \neq m$, i.e. $l \nmid m$, $\theta \in \mathrm{Hom}(\mathrm{Gal}(L_{ml^\infty}/L_{m_1}), \mathbb{Z}_l)$ is the restriction of a (unique) $\theta \in \mathrm{Hom}(\mathrm{Gal}(L_{ml^\infty}/L_m), \mathbb{Z}_l)$, and the projection formula for the cup product shows that β_ϖ is induced by cup product

$$H^1(\mathcal{O}_{L_m}[\frac{1}{ml}], \mathbb{Q}_l(1)) = \mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes \mathbb{Q}_l \xrightarrow{\theta\cup} X_{\{v|ml\}} \otimes \mathbb{Q}_l = H^2(\mathcal{O}_{L_m}[\frac{1}{ml}], \mathbb{Q}_l(1))$$

over L_m. For any place v of L_m we have a commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{O}_{L_m}[\frac{1}{ml}], \mathbb{Q}_l(1)) & \xrightarrow{\theta\cup} & H^2(\mathcal{O}_{L_m}[\frac{1}{ml}], \mathbb{Q}_l(1)) \\ \downarrow & & \downarrow \\ H^1(L_{m,v}, \mathbb{Q}_l(1)) & \xrightarrow{\mathrm{res}_v(\theta)\cup} & H^2(L_{m,v}, \mathbb{Q}_l(1)) \cong \mathbb{Q}_l \end{array}$$

and for $u \in \mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes \mathbb{Q}_l$ the element $\mathrm{res}_v(\theta) \cup \mathrm{res}_v(u)$ can be computed by local class field theory. For $v \nmid l$ one finds (see [49][Ch.II, 1.4.2]) that

$$\mathrm{res}_v(\theta) \cup \mathrm{res}_v(u) = -\theta(\mathrm{Fr}_v)\,\mathrm{ord}_v(u) = \theta(\mathrm{Fr}_p^{-f_p})\,\mathrm{ord}_v(u) = c_p\,\mathrm{ord}_v(u)$$

and for $v \mid l$ one has by [49][Ch. II, 1.4.5]

$$\mathrm{res}_v(\theta) \cup \mathrm{res}_v(u) = c_l \log_l(\chi_{\mathrm{cyclo}}) \cup u_v = c_l \mathrm{Tr}_{L_{m,v}/\mathbb{Q}_l}(\log_l(u_v)).$$

□

We verify the assumptions of Lemma 5.7 and describe the bases β_1 and β_2. The torsion submodule of $U^\infty_{\{v|ml\}}$ is $\mathbb{Z}_l(1)$ and hence $H^1(\Delta) = U^\infty_{\{v|ml\},\mathfrak{q}_\chi} = M^1$ is free of rank one over $\Lambda_{\mathfrak{q}_\chi}$ if χ is even, and $H^1(\Delta) = M^1 = 0$ for χ odd. It remains to find a basis if χ is even. The vanishing of $P^\infty_{\mathfrak{q}_\chi}$ combined with Lemma 5.4 gives that $U^\infty_{\mathfrak{q}_\chi} = \Lambda_{\mathfrak{q}_\chi} \cdot (1-\gamma)^\epsilon \cdot \eta_{f_{\chi,0}}$ where $f_{\chi,0} \mid m_0$ is the prime-to-l-part of f_χ and $\epsilon = 1$ or 0 according to whether χ factors through the cyclotomic $\mathbb{Z}_l$-extension of $\mathbb{Q}$ or not. In this latter case, we either have $\chi = 1$ or the element $1 - \chi(\gamma)$ is a unit in $\Lambda_{\mathfrak{q}_\chi}$ and $\eta_1 = \eta_{f_{\chi,0}}$ will in fact be a basis. Combined with (5.12) we find a (nonsplit) exact sequence

$$0 \to U^\infty_{\mathfrak{q}_\chi} \to U^\infty_{\{v|ml\},\mathfrak{q}_\chi} \to Y^\infty_{\{v|l\},\mathfrak{q}_\chi} \to 0$$

where the last term is nonzero (and then isomorphic to the residue field $\mathbb{Q}_l(\chi)$) only for $\chi(l) = 1$.

The exact sequence (5.12) together with Lemma 5.4 and Lemma 5.11 below for χ odd (resp. the vanishing of $P^\infty_{q_\chi}$ for χ even) show that $P^\infty_{\{v|ml\},\mathfrak{q}_\chi} = 0$ for any χ. Hence the $\Lambda_{\mathfrak{q}_\chi}$-torsion submodule of $H^2(\Delta)$ is $X^\infty_{\{v|ml\},\mathfrak{q}_\chi}$ and M^2 is canonically isomorphic to $Y^\infty_{\{v|\infty\},\mathfrak{q}_\chi}$ if $m_0 \neq 1$ and to $X^\infty_{\{v|l\infty\},\mathfrak{q}_\chi}$ if $m_0 = 1$. However, if $\chi \neq 1$ we have an isomorphism $X^\infty_{\{v|l\infty\},\mathfrak{q}_\chi} = Y^\infty_{\{v|l\infty\},\mathfrak{q}_\chi} = Y^\infty_{\{v|\infty\},\mathfrak{q}_\chi}$. Summarizing we have

(5.25)

	M^1	M^2
χ even, $\chi(l) \neq 1$	$\Lambda_{\mathfrak{q}_\chi} \cdot \eta_{f_{\chi,0}}$	$\Lambda_{\mathfrak{q}_\chi} \cdot \sigma$
$\chi \neq 1$ even, $\chi(l) = 1$	$\Lambda_{\mathfrak{q}_\chi} \cdot \frac{\eta_{f_{\chi,0}}}{\varpi}$	$\Lambda_{\mathfrak{q}_\chi} \cdot \sigma$
$\chi = 1$, $m_0 \neq 1$	$\Lambda_{\mathfrak{q}_\chi} \cdot \eta_1$	$\Lambda_{\mathfrak{q}_\chi} \cdot \sigma$
$\chi = 1$, $m_0 = 1$	$\Lambda_{\mathfrak{q}_\chi} \cdot \eta_1$	$\Lambda_{\mathfrak{q}_\chi} \cdot (\sigma - \lambda)$
χ odd	0	0

where λ is the unique place of L_{l^∞} above l.

We now break down the discussion into the various cases listed in this table. In each case we use a basis of the $\mathbb{Q}_l(\chi)$-space $\mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$, the source of the maps β_ϖ and val, in order to understand the trivializations ϕ and ϕ_ϖ. Then we write down $\mathcal{L}$ in terms of β_1 and β_2 and apply Lemma 5.7.

The $\mathbb{Q}_l(\chi)$-spaces $Y_{\{v|p\}} \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$ are nonzero if and only if $\chi(p) = 1$, in which case they are of dimension 1 with basis any fixed place $v_p \mid p$. For each such p let $x_p \in L_m$ be an element with nontrivial divisor concentrated at v_p. The $\mathbb{Q}_l(\chi)$-space spanned by x_p is then mapped isomorphically to $Y_{\{v|p\}} \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$ under the valuation map val. Put $J = \{p \mid m_0,\, \chi(p) = 1\}$, $x_J = \bigwedge_{p\in J} x_p$, $v_J = \bigwedge_{p\in J} v_p$ and $c_\chi = \prod_{p\in J} c_p$.

The case of even χ with $\chi(l) \neq 1$. The element $\bar\beta_1$ is the image of the norm compatible system

$$\eta_{f_{\chi,0}} = (1-\zeta_{f_{\chi,0}l^\nu}) \in \varprojlim_\nu \mathcal{O}_{L_{m_0l^\nu}}[\frac{1}{ml}]^\times \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

in $M^1/\varpi \subseteq \mathcal{O}_L[\frac{1}{ml}]^\times \otimes_{\mathbb{Z}[G]} \mathbb{Q}_l(\chi)$ where $L \subset L_{ml^\infty}$ is any field so that χ factors through $G = \mathrm{Gal}(L/\mathbb{Q})$. We insist on taking $L = L_m$ so that we have

$$\bar\beta_1 = \begin{cases} (1-\chi^{-1}(l))(1-\zeta_{f_{\chi,0}}) = (1-\chi^{-1}(l))(1-\zeta_{f_\chi}) & \mu = 0 \\ (1-\zeta_{f_{\chi,0}l^\mu}) = [L_m : L_{m_0l^{\mu'}}]^{-1}(1-\chi^{-1}(l))(1-\zeta_{f_\chi}) & \mu > 0. \end{cases}$$

where $\mu = \mathrm{ord}_l(m)$, $\mu' = \mathrm{ord}_l(f_\chi)$ and we recall the convention that $\chi^{-1}(l) = 0$ if $\mu' > 0$. The element $\bar\beta_1$ generates the one-dimensional subspace $M^1/\varpi \subseteq \mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$ of universal norms which coincides with $\mathcal{O}_{L_m}^\times \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$. The set $\{\bar\beta_1\} \cup \{x_p | \, p \in J\}$ is a basis of $\mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$ and we also have

$$\bar\beta_2 = \bar\sigma = \sigma_m \in Y_{\{v|\infty\}} \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi) = X_{\{v|\infty\}} \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi).$$

Hence by Lemma 5.8

$$\begin{aligned} &\phi \circ \phi_\varpi^{-1}(\bar\beta_1^{-1} \otimes \bar\beta_2) \\ &= \phi(\bar\beta_1^{-1} \wedge x_J^{-1} \otimes \beta_\varpi(x_J) \wedge \bar\beta_2) \\ &= c_\chi \phi(\bar\beta_1^{-1} \wedge x_J^{-1} \otimes \mathrm{val}(x_J) \wedge \bar\beta_2) \\ &= c_\chi [L_m : L_{m_0l^{\mu'}}](1-\chi^{-1}(l))^{-1} \ \phi\left([1-\zeta_{f_\chi}]^{-1} \wedge x_J^{-1} \otimes \mathrm{val}(x_J) \wedge \sigma_m\right) \\ &= c_\chi [L_m : L_{m_0l^{\mu'}}](1-\chi^{-1}(l))^{-1} \ [1-\zeta_{f_\chi}]^{-1} \otimes \sigma_m. \end{aligned} \tag{5.26}$$

As elements of $(\mathrm{Det}_\Lambda \Delta^\infty)_{\mathfrak{q}_\chi}$ we have using (5.15)

$$\begin{aligned} \mathcal{L} &= 2\cdot \eta_{m_0}^{-1} \otimes \sigma = 2 \cdot [L_{m_0} : L_{f_{\chi,0}}] \prod_{p|m_0,\, p\nmid f_{\chi,0}} \frac{1}{1-\mathrm{Fr}_p^{-1}} \cdot \eta_{f_{\chi,0}}^{-1} \otimes \sigma \\ &= 2 \cdot [L_{m_0} : L_{f_{\chi,0}}] \prod_{\substack{p|m_0,\, p\nmid f_{\chi,0} \\ \chi(p)\neq 1}} \frac{1}{1-\mathrm{Fr}_p^{-1}} \cdot \prod_{p\in J} \frac{\varpi}{1-\mathrm{Fr}_p^{-1}} \cdot \varpi^e \beta_1^{-1} \otimes \beta_2. \end{aligned} \tag{5.27}$$

For $p \in J$ we have $\mathrm{Fr}_p^{-f_p} = \gamma^{c_pl^n}$ and

$$\chi(\frac{\varpi}{1-\mathrm{Fr}_p^{-1}}) = \chi\left(\frac{(1+\mathrm{Fr}_p^{-1}+\cdots \mathrm{Fr}_p^{-f_p-1})(1-\gamma^{l^n})}{1-\gamma^{c_pl^n}}\right) = \frac{f_p}{c_p} \tag{5.28}$$

since $\chi(p) = \chi(\gamma^{l^n}) = 1$. Hence we conclude from Lemma 5.7, (5.26) and the identity $[L_m : L_{f_\chi}] = [L_{m_0} : L_{f_{\chi,0}}] \cdot [L_m : L_{m_0 l^{\mu'}}]$ that

$$\begin{aligned}\phi(\mathcal{L} \otimes 1) =& 2 \cdot [L_{m_0} : L_{f_{\chi,0}}] \prod_{\substack{p \mid m_0 \\ \chi(p) \neq 1}} \frac{1}{1 - \chi^{-1}(p)} \cdot \prod_{p \in J} \frac{f_p}{c_p} \cdot \phi \circ \phi_{\varpi}^{-1}(\bar{\beta}_1^{-1} \otimes \bar{\beta}_2) \\ =& 2 \cdot [L_m : L_{f_\chi}] \prod_{\substack{p \mid ml \\ \chi(p) \neq 1}} \frac{1}{1 - \chi^{-1}(p)} \cdot \prod_{p \in J} f_p \cdot [1 - \zeta_{f_\chi}]^{-1} \otimes \sigma_m \\ =& 2 \cdot [L_m : L_{f_\chi}] \prod_{p \mid ml} (\mathcal{E}_p^{\#}) \cdot [1 - \zeta_{f_\chi}]^{-1} \otimes \sigma_m\end{aligned}$$

which is the identity (5.20) to be shown.

The case of even $\chi \neq 1$ with $\chi(l) = 1$. In this case the subspace M^1/ϖ of universal norms in $\mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$ does not coincide with $\mathcal{O}_{L_m}^\times \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$. A basis of this latter space is given by $(1 - \zeta_{f_\chi})$ and a basis of $\mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$ is $\{\bar{\beta}_1, (1 - \zeta_{f_\chi})\} \cup \{x_p \mid p \in J\}$. The next Lemma gives the necessary information about $\bar{\beta}_1$.

LEMMA 5.10. *(Solomon) Let $L/\mathbb{Q}$ be an abelian extension of conductor $d > 1$ in which l splits completely, and denote by $L^{(\infty)}/L$ the cyclotomic $\mathbb{Z}_l$-extension and by $L^{(i)}/L$ the subfield of degree l^i. Then there is an element $\kappa(L^{(\infty)}, \gamma^{l^n}) = (\kappa(L^{(i)}, \gamma^{l^n}))_i \in \varprojlim_i \mathcal{O}_{L^{(i)}}[\frac{1}{l}] \otimes_{\mathbb{Z}} \mathbb{Z}_l$ so that $(1 - \gamma^{l^n}) \cdot \kappa(L^{(\infty)}, \gamma^{l^n}) = N \cdot \eta_d$ where $N \in \Lambda$ is the norm from $L_{d\ell l^n}$ to L, and*

$$\mathrm{ord}_w(\kappa(L, \gamma^{l^n})) = -c_l \log_l(\sigma_w(N_{L_d/L}(1 - \zeta_d))$$

for any place w of L dividing l.

PROOF. See [71]. □

We apply this Lemma to the splitting field L_χ of l in $L_{f_\chi}/\mathbb{Q}$ in which case $d = f_\chi = f_{\chi,0}$. We then have $\beta_1 = \varpi^{-1}\eta_{f_{\chi,0}} = N^{-1}\kappa(L_\chi^{(\infty)}, \gamma^{l^n})$ and

$$[L_{l^n \ell f_\chi} : L_\chi]\bar{\beta}_1 = \overline{\kappa(L_\chi^{(\infty)}, \gamma^{l^n})} = \kappa(L_\chi, \gamma^{l^n})$$

as elements of $\mathcal{O}_{L_{m_1}}[\frac{1}{ml}]^\times \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$. Taking norms from L_{m_1} to L_m we have

$$[L_{l^\mu f_\chi} : L_\chi]\bar{\beta}_1 = \overline{\kappa(L_\chi^{(\infty)}, \gamma^{l^n})} = \kappa(L_\chi, \gamma^{l^n})$$

in $\mathcal{O}_{L_m}[\frac{1}{ml}]^\times \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$ where $\mu = \mathrm{ord}_l(m)$. For each place $v \mid l$ of L_m denote by w the place of L_χ induced by v. By Lemma 5.10 we have

$$\mathrm{ord}_v(\kappa(L_\chi, \gamma^{l^n})) = |I_l| \, \mathrm{ord}_w(\kappa(L_\chi, \gamma^{l^n})) = -|I_l| c_l \log_l(N_{L_{f_\chi}/L_\chi}(1 - \zeta_{f_\chi})_v)$$

and hence

$$\begin{aligned}\mathrm{T}r_{L_{m,v}/\mathbb{Q}_l}(\log_l(1 - \zeta_{f_\chi})_v) =& [L_{m,v} : L_{f_\chi,v}] \log_l(N_{L_{f_\chi}/L_\chi}(1 - \zeta_{f_\chi})_v) \\ =& \frac{|D_l|}{[L_{f_\chi} : L_\chi]} \log_l(N_{L_{f_\chi}/L_\chi}(1 - \zeta_{f_\chi})_v) \\ =& -\frac{f_l}{c_l \cdot [L_{f_\chi} : L_\chi]} \mathrm{ord}_v(\kappa(L_\chi, \gamma^{l^n}))\end{aligned}$$

and therefore

$$\begin{aligned}\beta_\varpi(1-\zeta_{f_\chi}) =& c_l\sum_{v|l}\mathrm{Tr}_{L_{m,v}/\mathbb{Q}_l}(\log_l(1-\zeta_{f_\chi})_v)\cdot v\\ =& -\frac{f_l}{[L_{f_\chi}:L_\chi]}\sum_{v|l}\mathrm{ord}_v(\kappa(L_\chi,\gamma^{l^n}))\cdot v.\end{aligned}$$

Since $[L_{l^\mu f_\chi}:L_\chi]\bar\beta_1=\overline{\kappa(L_\chi^{(\infty)},\gamma^{l^n})}=\kappa(L_\chi,\gamma^{l^n})$ we conclude

$$\beta_\varpi(1-\zeta_{f_\chi})=-\frac{f_l\cdot[L_{l^\mu f_\chi}:L_\chi]}{[L_{f_\chi}:L_\chi]}\mathrm{val}(\bar\beta_1)=-f_l[L_{l^\mu f_\chi}:L_{f_\chi}]\,\mathrm{val}(\bar\beta_1).$$

We are now in a position to compute $\phi\circ\phi_\varpi^{-1}$.

$$\begin{aligned}&\phi\circ\phi_\varpi^{-1}(\bar\beta_1^{-1}\otimes\bar\beta_2)\\ =&\phi(\bar\beta_1^{-1}\wedge[1-\zeta_{f_\chi}]^{-1}\wedge x_J^{-1}\otimes\beta_\varpi(x_J)\wedge\beta_\varpi(1-\zeta_{f_\chi})\wedge\bar\beta_2)\\ =&-c_\chi f_l[L_{l^\mu f_\chi}:L_{f_\chi}]\ \phi(\bar\beta_1^{-1}\wedge[1-\zeta_{f_\chi}]^{-1}\wedge x_J^{-1}\otimes\mathrm{val}(x_J)\wedge\mathrm{val}(\bar\beta_1)\wedge\bar\beta_2)\\ =&c_\chi f_l[L_{l^\mu f_\chi}:L_{f_\chi}]\ \phi([1-\zeta_{f_\chi}]^{-1}\wedge\bar\beta_1^{-1}\wedge x_J^{-1}\otimes\mathrm{val}(x_J)\wedge\mathrm{val}(\bar\beta_1)\wedge\bar\beta_2)\\ =&c_\chi f_l[L_{l^\mu f_\chi}:L_{f_\chi}]\ [1-\zeta_{f_\chi}]^{-1}\otimes\sigma_m.\end{aligned}\tag{5.29}$$

The element $\mathcal{L}$ can be described as in (5.27) with only the power of ϖ changing. Combining this description with (5.28), (5.29), the identity $[L_m:L_{f_\chi}]=[L_{m_0}:L_{f_{\chi,0}}][L_{l^\mu f_\chi}:L_{f_\chi}]$ and Lemma 5.7 we obtain as before

$$\begin{aligned}\phi(\mathcal{L}\otimes 1)=&2\cdot[L_m:L_{f_\chi}]\prod_{\substack{p|m_0\\ \chi(p)\neq 1}}\frac{1}{1-\chi^{-1}(p)}\cdot\prod_{p\in J\cup\{l\}}f_p\cdot[1-\zeta_{f_\chi}]^{-1}\otimes\sigma_m\\ =&2\cdot[L_m:L_{f_\chi}]\prod_{p|ml}(\mathcal{E}_p^{\#})\cdot[1-\zeta_{f_\chi}]^{-1}\otimes\sigma_m\end{aligned}$$

which is the identity (5.20) to be shown.

The case of the trivial character. As in the discussion of the case of even χ with $\chi(l)\neq 1$ we first compute the element $\bar\beta_1\in\mathcal{O}_{L_m}[\frac{1}{ml}]^\times\otimes_{\mathfrak{A}}\mathbb{Q}_l$ (recall $\beta_1=\eta_1$) using the fact that $N_{L_\ell/\mathbb{Q}}(1-\zeta_\ell)=l$. We have

$$\bar\beta_1=\begin{cases}l & \mu=0\\ (1-\zeta_{l^\mu})=[L_m:L_{m_0}]^{-1}l & \mu>0.\end{cases}$$

A basis of $\mathcal{O}_{L_m}[\frac{1}{ml}]^\times\otimes_{\mathfrak{A}}\mathbb{Q}_l$ is given by $\{\bar\beta_1\}\cup\{x_p|\,p\in J\}=\{\bar\beta_1\}\cup\{x_p|\,p\mid m_0\}$. The map val is an isomorphism but β_ϖ is not. If $m_0>1$ a lift of $\bar\beta_2=\sigma_m$ to $X_{\{v|ml\infty\}}$ is given by σ_m-v_l where v_l is some place of L_m dividing l, and this remains true in the case $m_0=1$ where $\beta_2=\sigma-\lambda$. Moreover,

$$\mathrm{val}(l)=\sum_{v|l}|I_l|v=\frac{[L_m:\mathbb{Q}]}{f_l}v_l\mapsto-\frac{[L_m:\mathbb{Q}]}{f_l}(\sigma_m-v_l)=-\frac{[L_m:\mathbb{Q}]}{f_l}\bar\beta_2$$

and hence $\mathrm{val}(\bar\beta_1) = -[L_{m_0}:\mathbb{Q}]f_l^{-1}\bar\beta_2$. Therefore

$$\begin{aligned}\phi\circ\phi_\varpi^{-1}(\bar\beta_1^{-1}\otimes\bar\beta_2) =&\phi(\bar\beta_1^{-1}\wedge x_J^{-1}\otimes\beta_\varpi(x_J)\wedge\bar\beta_2)\\ =&-c_\chi\frac{f_l}{[L_{m_0}:\mathbb{Q}]}\phi(\bar\beta_1^{-1}\wedge x_J^{-1}\otimes\mathrm{val}(x_J)\wedge\mathrm{val}(\bar\beta_1))\\ =&-c_\chi\frac{f_l}{[L_{m_0}:\mathbb{Q}]}.\end{aligned}$$

Again we have

$$\mathcal{L} = 2\cdot[L_{m_0}:\mathbb{Q}]\prod_{p|m_0}\frac{\varpi}{1-\mathrm{Fr}_p^{-1}}\cdot\varpi^e\beta_1^{-1}\otimes\beta_2$$

and

$$\begin{aligned}\phi(\mathcal{L}\otimes 1) =&-2\cdot[L_{m_0}:\mathbb{Q}]\prod_{p|m_0}\frac{f_p}{c_p}\cdot c_\chi\cdot\frac{f_l}{[L_{m_0}:\mathbb{Q}]}\\ =&-2\cdot\prod_{p|ml}(\mathcal{E}_p^{\#}) = \zeta(0)^{-1}\cdot\prod_{p|ml}(\mathcal{E}_p^{\#})\end{aligned}$$

which is the identity (5.20).

The case of odd χ. In this case the maps β_ϖ and val are isomorphisms. The valuation map $u\mapsto\mathrm{val}(u)=\sum_{v|ml}\mathrm{ord}_v(u)\cdot v$ has a diagonal matrix in the bases $\{x_p\}$ and $\{v_p\}$. The matrix of the map β_ϖ on the other hand is upper triangular with diagonal terms

$$\beta_\varpi(x_p) = c_p\,\mathrm{val}(x_p);\quad \beta_\varpi(x_l) = c_l\sum_{v|l}\mathrm{Tr}_{L_{m,v}/\mathbb{Q}_l}(\log_l(x_{l,v}))\cdot v$$

where the term corresponding to p only occurs for $\chi(p)=1$. If $\chi(l)=1$ then we may pick x_l to lie in the splitting field of l in $L_m/\mathbb{Q}$ and we obtain

$$\sum_{v|l}\mathrm{Tr}_{L_{m,v}/\mathbb{Q}_l}(\log_l(x_{l,v}))\cdot v = |D_l|\sum_{g\in G_m/D_l}\log_l(\sigma_{v_l}(gx_l))\cdot g^{-1}v_l$$

where σ_{v_l} is the embedding corresponding to v_l (so that we have $\sigma_{v_l}(x_l)\in\mathbb{Q}_l$). The image of this element in $Y_{\{v|l\}}\otimes_{\mathfrak{A}}\mathbb{Q}_l(\chi)$ is

$$|D_l|\sum_{g\in G_m/D_l}\log_l(\sigma_{v_l}(gx_l))\cdot\chi(g)^{-1}v_l = |D_l|\sum_{g\in G_\chi}\log_l(\sigma_{v_l}(g\tilde x_l))\cdot\chi(g)^{-1}v_l$$

where $G_\chi=\mathrm{Gal}(L_\chi/\mathbb{Q})$ with L_χ the fixed field of χ and $\tilde x_l$ is the Norm of x_l into L_χ. We have $\mathrm{val}(x_l)=\mathrm{ord}_{v_l}(x_l)\cdot v_l = |I_l|\,\mathrm{ord}_{w_l}(\tilde x_l)\cdot v_l$ where w_l is the place of L_χ induced by v_l and $|I_l|$ is the ramification degree of l in $L_m/\mathbb{Q}$. Hence

$$\beta_\varpi(x_l) = c_l\frac{|D_l|\sum_{g\in G_\chi}\log_l(\sigma_{v_l}(g\tilde x_l))\cdot\chi(g)^{-1}}{|I_l|\,\mathrm{ord}_{w_l}(\tilde x_l)}\,\mathrm{val}(x_l)$$

and the map $\phi\circ\phi_\varpi^{-1}$ is just multiplication with

$$\phi\circ\phi_\varpi^{-1} = c_\chi c_l f_l\frac{\sum_{g\in G_\chi}\log_l(\sigma_{v_l}(g\tilde x_l))\cdot\chi(g)^{-1}}{\mathrm{ord}_{w_l}(\tilde x_l)}\tag{5.30}$$

where the last three factors only occur if $\chi(l) = 1$. We have $\beta_1 = \beta_2 = 1$ and in $(\mathrm{Det}_\Lambda \Delta^\infty)_{\mathfrak{q}_\chi}$ using (5.16)

$$\mathcal{L} = \theta_{m_0}^{-1} = \frac{\varpi^\delta}{\theta_{f_{\chi,0}}} \prod_{\substack{p|m_0,\, p\nmid f_{\chi,0} \\ \chi(p)\neq 1}} \frac{1}{1-\mathrm{Fr}_p^{-1}} \cdot \prod_{p\in J} \frac{\varpi}{1-\mathrm{Fr}_p^{-1}} \cdot \varpi^e. \tag{5.31}$$

where $\delta = 1, 0$ according to whether $\chi(l) = 1$ or not. If $\chi(l) \neq 1$ we have

$$\chi(\theta_{f_{\chi,0}}) = L(\chi^{-1}, 0)(1 - \chi^{-1}(l)) \tag{5.32}$$

using (5.6) and the fact that the g_k satisfy Euler-system relations.

Lemma 5.11. *(Ferrero-Greenberg) If $\chi(l) = 1$ then*

$$\chi\left(\frac{\theta_{f_{\chi,0}}}{\varpi}\right) = L(\chi^{-1}, 0) c_l \frac{\sum_{g\in G_\chi} \log_l(\sigma_{v_l}(g\tilde{x}_l)) \cdot \chi(g)^{-1}}{\mathrm{ord}_{w_l}(\tilde{x}_l)} \tag{5.33}$$

Proof. Consider the l-adic L-function

$$L_l(\chi^{-1}\omega, s) = (\chi\hat{\chi}_{\mathrm{cyclo}}^s)(\theta_{f_{\chi,0}})$$

where ω is the Teichmueller character, s is a variable in $\mathbb{Z}_l$, $\hat{\chi}_{\mathrm{cyclo}} = \chi_{\mathrm{cyclo}}\omega^{-1} : G_{ml^\infty} \to \mathbb{Z}_l^\times$, and we extend continuous characters of G_{ml^∞} to algebra homomorphisms $\Lambda \to \mathbb{Q}_l$ in the usual way. In [33] a formula is given for the derivative $L_l'(\chi^{-1}\omega, 0)$. For a certain l-unit (Gaussian sum) $\gamma_1 \in L_{\ell f_\chi}$ one has

$$L_l'(\chi^{-1}\omega, 0) = \frac{d}{ds} L_l(\chi^{-1}\omega, s)\Big|_{s=0} = \sum_{g\in G_{f_\chi}/<l>} \chi^{-1}(g) \log_l(\sigma_{v_l}(g\gamma_1)).$$

On the other hand, Stickelberger's theorem says that

$$\mathrm{val}(\gamma_1) = \sum_{g\in G_{f_\chi}/<l>} \mathrm{ord}_{gv_l}(\gamma_1) \cdot gv_l = \sum_{\substack{c=1 \\ (c,f_\chi)=1}}^{f_\chi} \frac{c}{f_\chi} \cdot \tau_c^{-1} v_l$$

where val is the valuation map normalized for the field L_{f_χ}. After applying χ we find

$$\mathrm{val}(\gamma_1) = \sum_{c=1}^{f_\chi} \frac{c}{f_\chi} \chi^{-1}(c) \cdot v_l = -L(\chi^{-1}, 0) \cdot v_l.$$

On the one-dimensional $\mathbb{Q}_l(\chi)$-space $\mathcal{O}_{L_{\ell f_\chi}}[\frac{1}{l}]^\times \otimes_{\mathbb{Z}[G_{\ell f_\chi}]} \mathbb{Q}_l(\chi) \cong \mathcal{O}_{L_m}[\frac{1}{l}] \otimes_{\mathfrak{A}} \mathbb{Q}_l(\chi)$ we have the two $\mathbb{Q}_l(\chi) \cdot v_l$-valued linear forms β_ϖ and val whose ratio can be computed by evaluating on either the element x_l of γ_1

$$\frac{\sum_{g\in G_\chi} \log_l(\sigma_{v_l}(g\tilde{x}_l)) \cdot \chi(g)^{-1}}{\mathrm{ord}_{w_l}(\tilde{x}_l)} = \frac{\sum_{g\in G_{f_\chi}/<l>} \chi^{-1}(g) \log_l(\sigma_{v_l}(g\gamma_1))}{-L(\chi^{-1}, 0)}.$$

Now the function $h(s) := (\chi\hat{\chi}_{\mathrm{cyclo}}^s)(\varpi) = (\chi\hat{\chi}_{\mathrm{cyclo}}^s)(1 - \gamma^{l^n})$ also has a first order zero at $s = 0$ with derivative $h'(0) = -\log_l(\chi_{\mathrm{cyclo}}(\gamma^{l^n})) = -c_l^{-1}$ and

$$\chi\left(\frac{\theta_{f_{\chi,0}}}{\varpi}\right) = \frac{(\chi\hat{\chi}_{\mathrm{cyclo}}^s)(g_{f_{\chi,0}})}{(\chi\hat{\chi}_{\mathrm{cyclo}}^s)(\varpi)}\Bigg|_{s=0} = \frac{L_l'(0, \chi^{-1}\omega)}{h'(0)}$$

$$= L(\chi^{-1}, 0) c_l \frac{\sum_{g\in G_\chi} \log_l(\sigma_{v_l}(g\tilde{x}_l)) \cdot \chi(g)^{-1}}{\mathrm{ord}_{w_l}(\tilde{x}_l)}.$$

□

By Lemma 5.7 (ϖ^e is mapped to 1) together with (5.30)-(5.33) and (5.28) we find

$$\phi(\mathcal{L}\otimes 1)=\frac{1}{L(\chi^{-1},0)}\prod_{\substack{p|ml\\ \chi(p)\neq 1}}\frac{1}{1-\chi^{-1}(p)}\cdot\prod_{\substack{p|ml\\ \chi(p)=1}} f_p=(L(\chi,0)^{\#})^{-1}\prod_{p|ml}\mathcal{E}_p^{\#}$$

which is (5.20).

The descent argument for $j<0$. In this section we again fix $1<m\not\equiv 2 \mod 4$ and $M=h^0(\mathrm{Spec}(L_m))(j)$ with $j<0$, and we shall prove Conjecture 3 for M and $A=\mathbb{Q}[G_m]$ in a way which is completely parallel to the case $j=0$. Theorem 5.2 is the key ingredient and the remaining arguments are computational.

The sequence $\mathbf{Mot}_\infty$ is the $\mathbb{R}$-dual (with contragredient G_m-action) of the isomorphism

$$K_{1-2j}(\mathcal{O}_{L_m})\otimes_{\mathbb{Z}}\mathbb{R}\xrightarrow{-\rho_\infty}\left(\bigoplus_{\tau\in\mathcal{T}}\mathbb{C}/\mathbb{R}\cdot(2\pi i)^{1-j}\right)^+$$

where ρ_∞ is the Beilinson regulator map, $\mathcal{T}=\mathrm{Hom}(L_m,\mathbb{C})$ and the $\mathbb{R}$-dual of this last space is identified with $\ker(\alpha_M)=M^+_{B,\mathbb{R}}$ by taking invariants in the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant perfect pairing

$$\bigoplus_{\tau\in\mathcal{T}}\mathbb{R}\cdot(2\pi i)^j\times\bigoplus_{\tau\in\mathcal{T}}\mathbb{C}/\mathbb{R}\cdot(2\pi i)^{1-j}\to\bigoplus_{\tau\in\mathcal{T}}\mathbb{C}/2\pi i\cdot\mathbb{R}\xrightarrow{\Sigma}\mathbb{R}$$

induced by multiplication. Note that this pairing identifies the $\mathbb{Q}$-dual of $M_B=\bigoplus_{\tau\in\mathcal{T}}\mathbb{Q}\cdot(2\pi i)^j$ with $\bigoplus_{\tau\in\mathcal{T}}\mathbb{Q}\cdot(2\pi i)^{-j}\subseteq\bigoplus_{\tau\in\mathcal{T}}\mathbb{C}/\mathbb{R}\cdot(2\pi i)^{1-j}$. Defining

$$Y(j):=\left(\bigoplus_{\tau\in\mathcal{T}}\mathbb{Q}\cdot(2\pi i)^j\right)^+$$

we obtain an identification as in the case $j=0$

$$\begin{aligned}\Xi(_AM)^{\#}&=\mathrm{Det}_A^{-1}(K_{1-2j}(\mathcal{O}_{L_m})\otimes_{\mathbb{Z}}\mathbb{Q})\otimes_A\mathrm{Det}_A Y(-j)\\ &=\prod_{\substack{\chi(-1)\\=(-1)^j}}(K_{1-2j}(\mathcal{O}_{L_m})\otimes_{\mathfrak{A}}\mathbb{Q}(\chi))^{-1}\otimes_{\mathbb{Q}(\chi)}(Y(-j)\otimes_{\mathfrak{A}}\mathbb{Q}(\chi))\times\prod_{\substack{\chi(-1)\\=(-1)^{j+1}}}\mathbb{Q}(\chi).\end{aligned}$$

The formulas for $L^*(\eta,0)$ in section 5.1 generalize to $j\leq 0$ (see [75][Ch.5]).

$$L(\eta,j)=-\frac{B_{1-j,\eta}}{1-j}:=-\frac{f_\eta^{-j}}{1-j}\sum_{a=1}^{f_\eta}B_{1-j}(\frac{a}{f_\eta})\eta(a)\in\mathbb{Q}(\eta)$$

$$\frac{d}{ds}L(\eta,s)|_{s=j}=(-j)!\left(\frac{2\pi i}{f_\eta}\right)^j\frac{1}{2}\sum_{a=1}^{f_\eta}\mathrm{Li}_{1-j}(e^{2\pi i a/f_\eta})\eta(a)\quad\text{ if }\eta(-1)=(-1)^j$$

Here B_k is the k-th Bernoulli polynomial and $\mathrm{Li}_k(x)=\sum_{n=1}^\infty\frac{x^n}{n^k}$.

THEOREM 5.12. *(Beilinson/Huber-Wildeshaus) For integers $f \mid m, j \leq 0$ there is an element $\xi_f(j) \in K_{1-2j}(\mathcal{O}_{L_f}) \otimes_{\mathbb{Z}} \mathbb{Q}$ whose image under the regulator map is given by*

$$-\rho_\infty(\xi_f(j)) = (-j)!f^{-j} \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^\times} \mathrm{Li}_{1-j}(e^{2\pi i a/f})\tau_a^{-1}(\sigma_m)$$

and whose image in $H^1(L_m, \mathbb{Q}_l(1-j))$ under the étale Chern class map ρ_l^{et} is given by

$$\rho_l^{et}(\xi_f(j)) = \left(\sum_{\alpha^{l^r} = \zeta_f} (1-\alpha) \otimes (\alpha^f)^{\otimes(-j)} \right)_r$$

PROOF. See [46][Cor. 9.6, 9.7]. In the notation of [46] we have

$$\xi_f(j) = (-j)!f^{-j}\epsilon_{1-j}(\zeta_f).$$

In particular $\xi_f(0) = 1 - \zeta_f$. □

We find that the image of $e_\eta \xi_{f_\eta}(j)$ under ρ_∞ is $[L_m : L_{f_\eta}] \cdot 2 \cdot L'(\eta^{-1}, j)(2\pi i)^{-j} \cdot \sigma_m$ and hence that ${}_A\vartheta_\infty(L^*({}_AM, 0)^{-1}) = (L^*({}_AM, 0)^{-1})^\# {}_A\vartheta_\infty(1)$ has components

$${}_A\vartheta_\infty(L^*({}_AM,0)^{-1})_\chi = \begin{cases} 2 \cdot [L_m : L_{f_\chi}][\xi_{f_\chi}(j)]^{-1} \otimes (2\pi i)^{-j} \cdot \sigma_m & \chi(-1) = (-1)^j \\ \left(L(\chi, j)^\#\right)^{-1} & \chi(-1) = (-1)^{j+1}. \end{cases}$$

Defining $\Delta(L_m)(j) = R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], T_l)^*[-3]$ we have isomorphisms

$$H^1(\Delta(L_m)(j))_{\mathbb{Q}_l} \cong H^1(\mathcal{O}_{L_m}[\frac{1}{ml}], \mathbb{Q}_l(1-j)) \xleftarrow{\rho_l^{et}} K_{1-2j}(\mathcal{O}_{L_m}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$$

$$H^2(\Delta(L_m)(j))_{\mathbb{Q}_l} \cong \left(\bigoplus_{\tau \in \mathcal{T}} \mathbb{Q}_l(-j) \right)^+ \cong Y(-j) \otimes_{\mathbb{Q}} \mathbb{Q}_l$$

The isomorphism ${}_A\vartheta_l : \Xi({}_AM)^\#_{\mathbb{Q}_l} \cong \mathrm{Det}_{A_l} \Delta(L_m)(j)_{\mathbb{Q}_l}$ sends ${}_A\vartheta_\infty(L^*({}_AM, 0)^{-1})$ to

(5.34)

$$\prod_{p \mid ml} \frac{1}{1 - \chi(p)^{-1}p^{-j}} \cdot 2 \cdot [L_m : L_{f_\chi}][\xi_{f_\chi}(j)]^{-1} \otimes \zeta_{l^\infty}^{\otimes -j} \cdot \sigma_m \qquad \text{if } \chi(-1) = (-1)^j$$

(5.35)

$$\prod_{p \mid ml} \frac{1}{1 - \chi(p)^{-1}p^{-j}} \cdot \left(L(\chi, j)^\#\right)^{-1} \qquad \text{if } \chi(-1) = (-1)^{j+1}$$

where ζ_{l^∞} is the generator of $\mathbb{Q}_l(1)$ given by the inverse system $(\zeta_{l^{n+1}})_{n \geq 0}$.

For $j \in \mathbb{Z}$ we denote by $\kappa^j : G_{ml^\infty} \to \Lambda^\times$ the character $g \mapsto \chi_{\mathrm{cyclo}}(g)^j g$ as well as the induced ring automorphism $\kappa^j : \Lambda \to \Lambda$. If there is no risk of confusion we also denote by $\kappa^j : \Lambda \to \mathfrak{A}_l \subseteq A_l$ the composite of κ^j and the natural projection to $\mathfrak{A}_l$ or A_l.

LEMMA 5.13. *a) For $j \in \mathbb{Z}$ there is a natural isomorphism*

$$\Delta^\infty \otimes^{\mathbb{L}}_{\Lambda, \kappa^j} \mathfrak{A}_l \cong \Delta(L_m)(j).$$

b) For $j \in \mathbb{Z}$ the image of an element

$$u = (u_n)_{n\geq 0} \in \varprojlim_n H^1(\mathcal{O}_{L_{m_0 l^n}}[\frac{1}{ml}], \mathbb{Z}/l^n\mathbb{Z}(1)) \cong U^\infty_{\{v|ml\}} = H^1(\Delta^\infty)$$

under the isomorphism $H^1(\Delta^\infty) \otimes_{\Lambda,\kappa^j} A_l \cong H^1(\mathcal{O}_{L_m}[\frac{1}{ml}], \mathbb{Q}_l(1-j))$ is given by

$$Tr_{L_{m_0 l^n}/L_m}(u_n \cup \zeta_{l^n}^{\otimes -j})_{n>>0} \tag{5.36}$$

c) For $j \in \mathbb{Z}$ the image of an element

$$s = (s_n)_{n\geq 0} \in \varprojlim_n \mathbb{Z}/l^n\mathbb{Z}[G_{m_0 l^n}] \cdot \sigma = Y^\infty_{\{v|\infty\}}$$

under the isomorphism $Y^\infty_{\{v|\infty\}} \otimes_{\Lambda,\kappa^j} A_l \cong H^0(\mathrm{Spec}(L_m \otimes \mathbb{R}), \mathbb{Q}_l(-j))$ is given by

$$(s_n \cup \zeta_{l^n}^{\otimes -j})_{n\geq 0}$$

Proof. The automorphism κ^j is the inverse limit of similarly defined automorphisms κ^j of the rings $\Lambda_n := \mathbb{Z}/l^n\mathbb{Z}[G_{m_0 l^n}]$. The sheaf $\mathcal{F}_n := f_{n,*}f_n^*\mathbb{Z}/l^n\mathbb{Z}$ (where $f_n : \mathrm{Spec}(\mathcal{O}_{L_{m_0 l^n}}[\frac{1}{ml}]) \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{ml}])$ is the natural map) is free of rank one over Λ_n with $\pi_1(\mathrm{Spec}(\mathbb{Z}[\frac{1}{ml}]))$-action given by the *inverse* of the natural projection $G_\mathbb{Q} \to G_{m_0 l^n}$. There is a Λ_n-κ^{-j}-semilinear isomorphism $\mathrm{tw}^j : \mathcal{F}_n \to \mathcal{F}_n(j)$. Shapiro's lemma gives a commutative diagram of isomorphisms

$$\begin{array}{ccc} R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], \mathcal{F}_n) & \xrightarrow{\mathrm{tw}^j} & R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], \mathcal{F}_n(j)) \\ \downarrow & & \downarrow \\ R\Gamma_c(\mathcal{O}_{L_{m_0 l^n}}[\frac{1}{ml}], \mathbb{Z}/l^n\mathbb{Z}) & \xrightarrow{\cup\zeta_{l^n}^{\otimes j}} & R\Gamma_c(\mathcal{O}_{L_{m_0 l^n}}[\frac{1}{ml}], \mathbb{Z}/l^n\mathbb{Z}(j)) \end{array} \tag{5.37}$$

where the horizontal arrows are Λ_n-κ^{-j}-semilinear. Taking the $\mathbb{Z}/l^n\mathbb{Z}$-dual of the lower row (with contragredient $G_{m_0 l^n}$-action) we obtain a $\# \circ \kappa^{-j} \circ \# = \kappa^j$-semilinear isomorphism

$$R\Gamma_c(\mathcal{O}_{L_{m_0 l^n}}[\frac{1}{ml}], \mathbb{Z}/l^n\mathbb{Z}(j))^*[-3] \to R\Gamma_c(\mathcal{O}_{L_{m_0 l^n}}[\frac{1}{ml}], \mathbb{Z}/l^n\mathbb{Z})^*[-3].$$

After passage to the limit this gives a κ^j-semilinear isomorphism $\Delta^\infty \cong \Delta^\infty(j)$, i.e. a Λ-linear isomorphism $\Delta^\infty \otimes_{\Lambda,\kappa^j} \Lambda \cong \Delta^\infty(j)$. Part a) follows by tensoring over Λ with $\mathfrak{A}_l$. The $\mathbb{Z}/l^n\mathbb{Z}$-dual of the H^2 of the inverse map in the lower row in (5.37) coincides with

$$H^1(\mathcal{O}_{L_{m_0 l^n}}[\frac{1}{ml}], \mathbb{Z}/l^n\mathbb{Z}(1-j)) \xleftarrow{\cup\zeta_{l^n}^{\otimes -j}} H^1(\mathcal{O}_{L_{m_0 l^n}}[\frac{1}{ml}], \mathbb{Z}/l^n\mathbb{Z}(1))$$

by Poitou-Tate duality. This gives b). Similarly to the lower row in (5.37) we have a κ^{-j}-semilinear map

$$\mathcal{F}_n^{c=1} = H^0(L_{m_0 l^n} \otimes \mathbb{R}, \mathbb{Z}/l^n\mathbb{Z}) \xrightarrow{\cup\zeta_{l^n}^{\otimes j}} H^0(L_{m_0 l^n} \otimes \mathbb{R}, \mathbb{Z}/l^n\mathbb{Z}(j)) = \mathcal{F}_n(j)^{c=1},$$

the $\mathbb{Z}/l^n\mathbb{Z}$-dual of which is the κ^j-semilinear isomorphism $\Lambda_n \cdot \sigma \leftarrow \Lambda_n \cdot \sigma \cup \zeta_{l^n}^{\otimes -j}$ given by cup product with $\zeta_{l^n}^{\otimes -j}$. Passing to the limit and tensoring over Λ with A_l we deduce c). □

By Lemma 5.13 a) we have an isomorphism of perfect complexes of $\mathfrak{A}_l$-modules

$$(\mathrm{Det}_\Lambda \Delta^\infty) \otimes_{\Lambda,\kappa^j} \mathfrak{A}_l \cong \mathrm{Det}_{\mathfrak{A}_l} \Delta(L_m)(j)$$

under which $\mathcal{L} \otimes 1$ is an $\mathfrak{A}_l$-basis of the right hand side by Theorem 5.2. For any $\chi \in \hat{G}_m^{\mathbb{Q}_l}$ we have the corresponding ring homomorphism

$$\chi\kappa^j : \Lambda \to \mathbb{Q}_l(\chi)$$

whose kernel we denote by $\mathfrak{q}_{\chi,j}$. This is a regular prime of Λ and we again apply Lemma 5.7 with $R = \Lambda_{\mathfrak{q}_{\chi,j}}$.

If $\chi(-1) = (-1)^{j+1}$ then $\psi_{\mathfrak{q}_{\chi,j}}(-1) = \chi(-1)\omega(-1)^j = -1$ i.e. $\psi_{\mathfrak{q}_{\chi,j}}$ is odd. In this case $\Delta^\infty_{\mathfrak{q}_{\chi,j}}$ is acyclic (there are no trivial zeros at the character $\chi\kappa^j$) and Lemma 5.7 applies with $\beta_i = b = 1$. The image of $\mathcal{L}$ in $(\mathrm{Det}_\Lambda \Delta^\infty)_{\mathfrak{q}_{\chi,j}}$ is

$$\mathcal{L} = \theta_{m_0}^{-1} = \theta_{f_{\chi,0}}^{-1} \prod_{p|m_0, p\nmid f_{\chi,0}} \frac{1}{1 - \mathrm{Fr}_p^{-1}}$$

and the image of $\mathcal{L} \otimes 1$ is

$$\chi\kappa^j(\mathcal{L}) = \prod_{p|m_0} \frac{1}{1-\chi(p)^{-1}p^{-j}} \, L_l(\chi^{-1}\omega^{1-j}, j)^{-1} \tag{5.38}$$

$$= \prod_{p|ml} \frac{1}{1-\chi(p)^{-1}p^{-j}} \, L(\chi^{-1}, j)^{-1}$$

where the last equality follows from (5.7). This value agrees with (5.35) which finishes the proof of Conjecture 3 in the case $\chi(-1) = (-1)^{j+1}$.

If $\chi(-1) = (-1)^j$ then the character $\psi_{\mathfrak{q}_{\chi,j}}$ is even and in order to apply Lemma 5.7 with $R = \Lambda_{\mathfrak{q}_{\chi,j}}$ we must describe the bases β_1 and β_2 in this case. By Lemma 5.13 b) the image of $\eta_{f_{\chi,0}} \in H^1(\Delta^\infty)_{\mathfrak{q}_{\chi,j}}$ in $M^1/\varpi \cong H^1(\mathcal{O}_{L_m}[\frac{1}{ml}], \mathbb{Q}_l(1-j))$ is given by the element described in Theorem 5.12, in particular it is nonzero. Hence $\beta_1 := \eta_{f_{\chi,0}}$ is a $\Lambda_{\mathfrak{q}_{\chi,j}}$-basis of M_1, and by Lemma 5.4 the image of σ is then a basis of $M^2 = H^2(\Delta^\infty)_{\mathfrak{q}_{\chi,j}}$. By Lemma 5.13 c) the image of σ in M^2/ϖ is $\zeta_{l^\infty}^{\otimes -j} \cdot \sigma_m$. The computation showing that $\mathcal{L} \otimes 1$ equals the element in (5.34) is now exactly the same as in the case $j = 0$, χ even, $\chi(l) \neq 1$, $J = \emptyset$.

5.2. CM elliptic curves and the main conjecture for imaginary quadratic fields. Let K be an imaginary quadratic field, and ψ a Hecke character of K of infinity type $(1,0)$ (such a ψ always exists; see [77] for a nice discussion of such characters with minimal conductor and field of values). The field of values of ψ is a CM-field and we denote by $\bar{\psi}$ the conjugate character. Any algebraic Hecke character of K is of the form $\Psi = \psi^k \bar{\psi}^j \chi$ where $j, k \in \mathbb{Z}$ and χ is a finite order (Dirichlet) character of K.

The computations of the previous section should have analogues for motives $M(\Psi)$ associated to any algebraic Hecke character Ψ of K, and all primes l but this has not been worked out in all cases. The analogue of the Iwasawa main conjecture (Theorem 5.2 above) is actually simpler since there is no distinction between even and odd characters: The l-adic L-function $\mathcal{L}$ is just given by an appropriate norm compatible system η_{m_0} of elliptic units. The method of Euler systems allows to prove this main conjecture (for l not dividing the number of roots so unity in the Hilbert class field of K a proof is given in [66]), and the analogue of the theorem of Ferrero and Washington on the vanishing of the μ-invariant is also known if l is split in $K/\mathbb{Q}$ [38].

The critical case. Recall that a motive M is called critical if $\ker(\alpha_M) = \operatorname{coker}(\alpha_M) = 0$. If the weight of M is different from -1 Conjecture Mot_∞ then implies that $H^i_f(M) = H^i_f(M^*(1)) = 0$ for all i. If $M(\Psi)$ is critical and of negative weight (modulo replacing Ψ by $\bar{\Psi}$ this is the range $k < 0$ and $0 \leq j < -k$) the relation of η_{m_0} to the leading coefficient $L^*(M(\Psi))$ is given by an explicit reciprocity law due to Wiles (for $k = -1, j = 0$ [78]), Kato (for $j = 0$ [49][Thm. 2.1.7]) and Tsuji (in general [74]). Proofs of Conjecture 3 (for certain critical Ψ and certain l and A) can be found in [66][Thm. 11.1], [43], [42], [26], [24]. We quote here one result from [24] dealing with the classical Birch and Swinnerton-Dyer case $k = -1, j = 0$ but with emphasis on non-maximal orders.

THEOREM 5.14. *(Colwell) Let F/K be an abelian extension and E/F an elliptic curve with CM by K and so that the Weil-Restriction $B = Res^F_K(E)$ is of CM-type. Assume* $\operatorname{rank}_{\mathbb{Z}} E(F) = 0$. *Then Conjecture 3 holds for $M = h^1(B)(1)$, $\mathfrak{A} = End_K(B)$ and $l > 3$ any prime number not dividing the class number of K.*

Note here that $\mathrm{End}_F(E)$ may be any order in K and that $\mathfrak{A}$ is usually a non-maximal order in $A = \mathrm{End}_K(B) \otimes \mathbb{Q}$.

The non-critical case. Here our knowledge is incomplete in one basic aspect: The $\mathbb{Q}$-space $H^1_f(M(\Psi))$ is not known to be finite dimensional unless $M(\Psi)$ is a direct summand of a motive discussed in our Example a) (i.e. $k = j$ and we are dealing with $M = h^0(\mathrm{Spec}(L))(-j)$ where L/K is an abelian extension). One actually works with an explicit subspace of $H^1_f(M(\Psi))$ of the expected dimension (i.e. such that Conjecture $\mathbf{Mot}_\infty$ holds). The construction of this space is due to Deninger [29] and the computation (in certain cases) of the étale regulator of its elements due to Kings. Conjecture $\mathbf{Mot}_l$ is not known for this space in all cases (this is equivalent to the vanishing of $H^2(\mathbb{Z}[\frac{1}{ml}], M_l)$ discussed in [53]). To illustrate we quote the main result from [53] corresponding to $k < -1, j = k + 1$.

THEOREM 5.15. *(Kings) Let E/K be an elliptic curve with CM by $\mathcal{O}_K$. Then Conjecture 3 holds for $M = h^1(E)(j)$ where $j \geq 2$, $\mathfrak{A} = \mathcal{O}_K$ and $l > 3$ is prime to the conductor of E and such that $H^2(\mathbb{Z}[\frac{1}{ml}], M_l) = 0$.*

It is quite likely that the arguments of Kings can be used to prove Conjecture 3 for $M = h^1(E)(j)$ with $j \leq 0$ *without* any assumption on a vanishing of H^2. For a generalisation of Kings' method to some other Hecke characters of K see [3].

5.3. Adjoint motives of modular forms. Let f be a holomorphic newform on the upper half plane (of weight $k \geq 2$, level N and some character), denote by E_f the number field generated by the Fourier-coefficients of f and by $M(f)$ the motive associated to f (of rank 2 over E_f). Let S_f be the finite set of places λ of E_f which either divide $Nk!$ or such that the G_F-representation on T_λ/λ is absolutely reducible, where F is the quadratic subfield of $\mathbb{Q}(\zeta_l)$, l is the rational prime below λ, and $T_\lambda \subset M(f)_\lambda$ is a $G_{\mathbb{Q}}$-stable lattice.

THEOREM 5.16. *(Diamond, Flach, Guo [31]) Let $M = Ad^0 M(f)$ be the adjoint motive of $M(f)$ consisting of all endomorphisms of trace 0 in* $\mathrm{Hom}(M(f), M(f))$. *Then conjecture 3 holds for M or $M(1)$, $\mathfrak{A} = \mathcal{O}_{E_f}$ and any prime $\lambda \notin S_f$.*

The proof is rather different from the previous examples as it is not based on either Euler systems or an Iwasawa Main Conjecture. The key ingredient is the Taylor-Wiles method developed to show modularity of elliptic curves over $\mathbb{Q}$.

For an integer N let Σ_N be the set of newforms of weight 2, level N and trivial character. For the adjoint of the motive $M := h^1(X_0(N)) \cong \prod_{f\in\Sigma_N} M(f)$ Theorem 5.16 implies Conjecture 3 with respect to the maximal order $\mathfrak{A} = \prod_{f\in\Sigma_N} \mathcal{O}_{E_f}$ in the algebra $A = \prod_{f\in\Sigma_N} E_f \cong \mathrm{End}(J_0(N)) \otimes \mathbb{Q}$. This can be refined as follows.

THEOREM 5.17. *(Qiang Lin [58]) Let N be a prime number, $M = Ad^0 h^1(X_0(N))$ and $\mathfrak{T}_N$ the integral Hecke algebra of weight 2 and level N (which is known to coincide with $\mathrm{End}(Jac(X_0(N)))$). Then Conjecture 3 holds for M or $M(1)$, $\mathfrak{A} = \mathfrak{T}_N$ and primes l so that $\lambda \notin S_f$ for all $\lambda \mid l$ and all $f \in \Sigma_N$.*

This is proven by an explicit algebraic computation using Theorem 5.16 and the fact that $\mathfrak{A}[\frac{1}{2}]$ is a local complete intersection ring (which follows from the Taylor-Wiles method).

5.4. Motives of modular forms. The results for motives $M(f)$ associated to newforms are less complete than those discussed in the previous sections. For example, Conjecture 3 includes the conjecture of Birch and Swinnerton-Dyer (BSD) for (modular) elliptic curves E over $\mathbb{Q}$. In the last two decades there has been quite some progress regarding this conjecture in the case $\mathrm{ord}_{s=1} L(E,s) \leq 1$ (but none for higher vanishing order). Both the Euler system of Heegner points, discovered by Kolyvagin [55], and the Euler system of K_2-elements, discovered by Kato [50], allow to prove upper bounds for the Tate-Shafarevich group which are related to $L^*(E,1)$. Sometimes this suffices to prove equality, either because the upper bound is 1, as it is for all but finitely many l, or because it can be achieved by the construction of elements in the Tate-Shafarevich group (for example using the idea of visibility due to Mazur and Cremona [25]). This allows to verify the l-primary part of the BSD-conjecture for many curves E and primes l (but eventually only for E in a finite list of examples). The paper [2] contains an extension of visibility arguments to modular abelian varieties of dimension > 1, corresponding to weight 2 forms f with $E_f \neq \mathbb{Q}$.

We remark at this point that a strategy for proving Kato's Iwasawa main conjecture [50][Conj. 17.6] for the cyclotomic deformation of motives $M(f)$ at ordinary primes l has been outlined recently by Skinner and Urban. This main conjecture is an analogue of Theorem 5.2, and descent computations along the lines of those given after Lemma 5.7 allow to deduce the l-part of the conjecture of Birch and Swinnerton-Dyer if $\mathrm{ord}_{s=1} L(E,s) \leq 1$. These computations are straightforward if $\mathrm{ord}_{s=1} L(E,s) = 0$ but require an l-adic analogue of the Gross-Zagier formula due to Perrin-Riou [61] if $\mathrm{ord}_{s=1} L(E,s) = 1$.

In a similar vein, Bertolini and Darmon [8] study the main conjecture for the anticyclotomic deformation of motives $M(f)$ (at least if f is attached to an elliptic curve E) with respect to an auxiliary imaginary quadratic field K and an ordinary prime l. This line of research might eventually lead to cases of the Birch and Swinnerton-Dyer conjecture for E over ring class fields H of K (and $\mathfrak{A} = \mathbb{Z}[\mathrm{Gal}(H/K)]$).

The motive $M = h^1(X_0(N))(1)$. A reader familiar with the usual formulation of the BSD-conjecture might not recognize it in Conjecture 3. In this section we illustrate how to go back and forth between the two formulations in the example $M = h^1(X_0(N))(1)$, N prime, $\mathfrak{A} = \mathfrak{T}_N := \mathrm{End}(J_0(N))$ (the adjoint of which was considered in Theorem 5.17). We hope that such a reformulation can serve as a basis for numerical checks of the equivariant conjecture over $\mathfrak{T}_N$, in the spirit of the paper [2] by Stein and Agashe.

In any case the link between Conjecture 3 and its classical counterpart (i.e. the formulation originally used by Bloch and Kato in [10]) is given by an integral version

$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l) \to R\Gamma_f(\mathbb{Q}, T_l) \to \bigoplus_{p\in S} R\Gamma_f(\mathbb{Q}_p, T_l) \tag{5.39}$$

of the exact triangle (3.1). One has to ensure that all terms are perfect complexes of $\mathfrak{A}_l$-modules and there may not be a natural (or any) way to do so. If $\mathfrak{A} = \mathbb{Z}$ however, or more generally if $\mathfrak{A}$ is maximal and hence a product of Dedekind rings, one can simply define $R\Gamma_f(\mathbb{Q}_p, T_l)$ by prescribing its cohomology to be the group $H^1_f(\mathbb{Q}_p, T_l) \subseteq H^1_f(\mathbb{Q}_p, M_l)$ defined by Bloch and Kato in [10] (see the end of section 1.5 in [14]).

Coming back to our example where $\mathfrak{A} = \mathfrak{T}_N$ we first recall Mazur's fundamental result [60] that for $l \neq 2$ the module $T_l := H^1(X_0(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_l(1))$ is free of rank 2 over $\mathfrak{T}_{N,l}$. In particular the projectivity condition before Conjecture 3 is satisfied. Mazur also shows that $\mathcal{D}_l := H^1_{dR}(X_0(N)/\mathbb{Z}_l)(1)$ and $\mathrm{Fil}^0\,\mathcal{D}_l = H^0(X_0(N)_{/\mathbb{Z}_l}, \Omega^1_{X_0(N)/\mathbb{Z}_l})$ are free of rank 2, resp. 1 over $\mathfrak{T}_{N,l}$. We shall only be able to exploit this in the "good reduction case" where $l \neq N$ although the following arguments might be pushed to $l = N$. We have $D_l := D_{cris}(M_l) = D_{dr}(M_l)$ and by [10][Lemma 4.5] there is a quasi-isomorphism (given by the vertical map)

$$\begin{array}{ccc} \mathrm{Fil}^0 D_l & \xrightarrow{1-\mathrm{Fr}_p} & D_l \\ \downarrow & & \downarrow \\ D_l & \xrightarrow{(1-\mathrm{Fr}_p,\pi)} & D_l \oplus D_l/\,\mathrm{Fil}^0 D_l. \end{array}$$

The top row contains an obvious integral complex, and we may define a perfect $\mathfrak{T}_{N,l}$-complex $R\Gamma_f(\mathbb{Q}_p, T_l)$ as

$$R\Gamma_f(\mathbb{Q}_p, T_l) = \begin{cases} T_l^{I_p} \xrightarrow{1-\mathrm{Fr}_p} T_l^{I_p} & l \neq p \\ \mathrm{Fil}^0\,\mathcal{D}_l \xrightarrow{1-\mathrm{Fr}_p} \mathcal{D}_l & l = p. \end{cases}$$

If we then define the complex $R\Gamma_f(\mathbb{Q}, T_l)$ by the exact triangle before (3.1) its cohomology is given as follows.

Lemma 5.18. *If* $Ш(J_0(N))$ *is finite then*

$$H^0_f(\mathbb{Q}, T_l) = 0, \quad H^3_f(\mathbb{Q}, T_l) \cong \mathrm{Hom}_{\mathbb{Z}}(J_0(N)(\mathbb{Q})_{l^\infty}, \mathbb{Q}_l/\mathbb{Z}_l)$$

and there are exact sequences of $\mathfrak{T}_{N,l}$*-modules*

$$0 \to H^1_f(\mathbb{Q}, T_l) \to J_0(N)(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \Phi_{N,l^\infty} \to H^2_f(\mathbb{Q}, T_l) \to H^2_f(\mathbb{Q}, T_l)^{BK} \to 0$$

$$0 \to Ш(J_0(N))_{l^\infty} \to H^2_f(F, T)^{BK} \to \mathrm{Hom}_{\mathbb{Z}}(J_0(N)(\mathbb{Q}), \mathbb{Z}_l) \to 0$$

where Φ_N is the group of components of the reduction of the Neron model of $J_0(N)$ in characteristic N.

Proof. See the computations in [14][(1.35)-(1.37)] □

For simplicity we now put ourselves in a rank zero situation. The winding quotient $\pi : J_0(N) \to J$ is an abelian variety over $\mathbb{Q}$, maximal (up to isogeny) with respect to the property that $L(h^1(J), 1) \neq 0$. More precisely, if $e \in H_1(X_0(N), \mathbb{Q})^+$ is the image of the path from $i\infty$ to 0 in the upper half plane, $\mathfrak{I}_e \subseteq \mathfrak{T}_N$ is the annihilator of e and

$$\mathfrak{I} = \mathrm{Ann}_{\mathfrak{T}_N}(\mathrm{Ann}_{\mathfrak{T}_N}(\mathfrak{I}_e)) \supseteq \mathfrak{I}_e$$

is the saturation of $\mathfrak{I}_e$ then $J = J_0(N)/\mathfrak{I}_e J_0(N) = J_0(N)/\mathfrak{I}J_0(N)$ and

$$H_1(J(\mathbb{C}), \mathbb{Z}) \cong H_1(J_0(N)(\mathbb{C}), \mathbb{Z})/\mathfrak{I} \cdot H_1(J_0(N)(\mathbb{C}), \mathbb{Z})$$

(after inverting 2). In particular, the homology of J is locally free (away from 2) over $\mathfrak{T} := \mathfrak{T}_N/\mathfrak{I}$ and the conditions before Conjecture 3 are satisfied for J and $\mathfrak{A} = \mathfrak{T}$ and $l \neq 2$. The usual BSD-conjecture for J is studied for example in [1] and we shall make precise the conjecture over $\mathfrak{A} = \mathfrak{T}$.

Mazur [60] shows that $\mathfrak{T}_N[\frac{1}{2}]$ is Gorenstein, and we shall henceforth assume the same for $\mathfrak{T}[\frac{1}{2}]$ (for many N it will simply be the case that $\mathfrak{I}[\frac{1}{2}] = (\frac{1+w}{2})$ where w is the Atkin Lehner involution, and $\mathfrak{T}[\frac{1}{2}]$ will be a direct factor of $\mathfrak{T}_N[\frac{1}{2}]$ and hence Gorenstein). Then the homology of the dual abelian variety $\hat{J}$ is also locally free over $\mathfrak{T}$. Denoting Neron models by $/\mathbb{Z}$, it is known that the map

$$\pi^* : H^0(J_{/\mathbb{Z}}, \Omega^1_{J/\mathbb{Z}}) \to H^0(J_0(N)_{/\mathbb{Z}}, \Omega^1_{J_0(N)/\mathbb{Z}})[\mathfrak{I}_e]$$

is injective and has 2-torsion cokernel (the cokernel may be regarded as a generalized Manin constant attach to J). Dualizing, we find that $H^1(\hat{J}_{/\mathbb{Z}_l}, \mathcal{O})$ is free of rank 1 over $\mathfrak{T}_l$ for $l \neq 2$ and so this also holds for $H^0(J_{/\mathbb{Z}_l}, \Omega^1_{J/\mathbb{Z}_l})$ and $H^1_{dR}(J/\mathbb{Z}_l)$ by our Gorenstein assumption. In summary, all of the above considerations apply to J or $\hat{J}$ in place of $J_0(N)$.

For $M = h^1(\hat{J})(1)$ we have $A := \mathfrak{T}_{\mathbb{Q}}$-isomorphisms

$$M_B = H^1(\hat{J}(\mathbb{C}), 2\pi i \cdot \mathbb{Q}) \cong H^1(J(\mathbb{C}), \mathbb{Q})^* \cong H_1(J(\mathbb{C}), \mathbb{Q})$$
$$M_{dR}/\operatorname{Fil}^0 M_{dR} \cong H^1(\hat{J}, \mathcal{O}_{\hat{J}}) \cong H^0(J, \Omega^1_{J/\mathbb{Q}})^*$$
$$\Xi({}_AM) = (H_1(J(\mathbb{C}), \mathbb{Q})^+)^{-1} \otimes_A H^0(J, \Omega^1_{J/\mathbb{Q}})^*$$

where $*$ denotes the $\mathbb{Q}$-dual. The period isomorphism sends a path γ to the linear form $\omega \mapsto \int_\gamma \omega$ on differentials, and $\alpha_M : H_1(J(\mathbb{C}), \mathbb{R})^+ \cong H^0(J, \Omega^1_{J/\mathbb{Q}})^*_{\mathbb{R}}$ is an isomorphism. By a well known fact in the theory of modular forms the $\mathbb{Z}$-linear form $a_1 \in H^0(J_0(N)_{/\mathbb{Z}}, \Omega^1_{J_0(N)/\mathbb{Z}})^*$ given by

$$H^0(J_0(N)_{/\mathbb{Z}}, \Omega^1_{J_0(N)/\mathbb{Z}}) \cong S^2(\Gamma_0(N), \mathbb{Z}) \ni g \mapsto a_1(g)$$

is in fact a $\mathfrak{T}_N$-basis of $H^0(J_0(N)_{/\mathbb{Z}}, \Omega^1_{J_0(N)/\mathbb{Z}})^*$ and hence its image is a $\mathfrak{T}$-basis of

$$H^0(J_0(N)_{/\mathbb{Z}}, \Omega^1_{J_0(N)/\mathbb{Z}})^*/\mathfrak{I} \cong H^0(J_{/\mathbb{Z}}, \Omega^1_{J/\mathbb{Z}})^*$$

away from 2.

Lemma 5.19. *With the notation introduced above we have*

$${}_A\vartheta_\infty^{-1}(L^*({}_AM, 0)^{-1}) = e^{-1} \otimes a_1 \in \Xi({}_AM)$$

PROOF. By definition of ${}_A\vartheta_\infty$ we have ${}_A\vartheta_\infty(e^{-1}\otimes\alpha_M(e))=1$. We can write $\alpha_M(e)=t\cdot a_1$ with $t=(t_f)\in\mathfrak{T}_\mathbb{R}\cong\prod_{f\in\Sigma'_N}\mathbb{R}$ where Σ'_N is the subset of forms in Σ_N corresponding to J. Applying this identity to a newform $f\in\Sigma'_N$ we find

$$L(f,1)=\int_{i\infty}^0 f(q)\frac{dq}{q}=\int_e\omega_f=\alpha_M(e)(\omega_f)=ta_1(\omega_f)=t_fa_1(f)=t_f.$$

Hence $t=L({}_AM,0)=L^*({}_AM,0)$ and ${}_A\vartheta_\infty(e^{-1}\otimes a_1)=L^*({}_AM,0)^{-1}$. □

One verifies easily that a_1 is in fact a $\mathfrak{T}_l$-basis of $\det_{\mathfrak{T}_l}R\Gamma_f(\mathbb{Q}_p,T_l)$ as defined above. However, e need not be a $\mathfrak{T}_l$-basis of $H_1(J(\mathbb{C}),\mathbb{Z}_l)^+$. So let $\mathfrak{b}$ be the invertible fractional $\mathfrak{T}$-ideal so that

$$\mathfrak{b}\cdot e=H_1(J(\mathbb{C}),\mathbb{Z})^+ \tag{5.40}$$

away from $l=2$. The triangle (5.39) induces an isomorphism

$$\begin{aligned}&\mathrm{Det}_{\mathfrak{T}_l}R\Gamma_c(\mathbb{Z}[\frac{1}{S}],T_l)\\ \cong&\mathrm{Det}_{\mathfrak{T}_l}R\Gamma_f(\mathbb{Q},T_l)\otimes\mathrm{Det}^{-1}_{\mathfrak{T}_l}R\Gamma_f(\mathbb{Q}_l,T_l)\otimes\mathrm{Det}^{-1}_{\mathfrak{T}_l}H_1(J(\mathbb{C}),\mathbb{Z})^+\\ \cong&\mathrm{Det}_{\mathfrak{T}_l}R\Gamma_f(\mathbb{Q},T_l)\otimes\mathfrak{b}^{-1}a_1\otimes e^{-1}.\end{aligned}$$

Now recall that since the rank of $J(\mathbb{Q})$ is zero $R\Gamma_f(\mathbb{Q},T_l)$ is a (perfect) $\mathfrak{T}_l$-complex with torsion cohomology and hence

$$\mathrm{Det}_{\mathfrak{T}_l}R\Gamma_f(\mathbb{Q},T_l)\subseteq\mathrm{Det}_{\mathfrak{T}_{\mathbb{Q}_l}}R\Gamma_f(\mathbb{Q},M_l)\cong\mathfrak{T}_{\mathbb{Q}_l}$$

identfies with a fractional ideal. The statement that $e^{-1}\otimes a_1$ is a basis of the invertible $\mathfrak{T}_l$-module $\mathrm{Det}_{\mathfrak{T}_l}R\Gamma_c(\mathbb{Z}[\frac{1}{S}],T_l)$ is equivalent to

$$\mathrm{Det}_{\mathfrak{T}_l}R\Gamma_f(\mathbb{Q},T_l)=\mathfrak{b}_l. \tag{5.41}$$

Summarizing we have

PROPOSITION 5.1. *Let N be a prime number and J the winding quotient for $J_0(N)$. For any prime $l\neq 2,N$ there exists a perfect complex $R\Gamma_f(\mathbb{Q},T_l)$ of $\mathfrak{T}_l$-modules with cohomology in degrees* 1, 2, 3 *given by the exact sequence*

$$0\to H^1_f(\mathbb{Q},T_l)\to J(\mathbb{Q})_{l^\infty}\to\Phi_{l^\infty}\to H^2_f(\mathbb{Q},T_l)\to Ш(J/\mathbb{Q})_{l^\infty}\to 0$$

and $H^3_f(\mathbb{Q},T_l)\cong\hat{J}(\mathbb{Q})_{l^\infty}$. Moreover, the l-primary part of Conjecture 3 *for $M=h^1(\hat{J})(1)$ and $\mathfrak{A}=\mathfrak{T}$ is equivalent to (5.41) where $\mathfrak{b}$ is defined in (5.40).*

Taking norms from $\mathfrak{T}$ to $\mathbb{Z}$ we find that the usual BSD-conjecture for J is equivalent to the statement

$$\frac{|Ш(J/\mathbb{Q})||\Phi|}{|J(\mathbb{Q})||\hat{J}(\mathbb{Q})|}=N_{\mathfrak{T}/\mathbb{Z}}(\mathfrak{b})^{-1}$$

up to powers of 2 and N. The right hand side is the index of the $\mathbb{Z}$-lattice $\mathfrak{T}\cdot e$ in the lattice $H_1(J(\mathbb{C}),\mathbb{Z})^+$ (in the generalized sense since the two are not contained in each other). We therefore (almost) recover Agashe's formula [1][Eq. (1)]. The difference is that we work with the quotient $H_1(J(\mathbb{C}),\mathbb{Z})$ of $H_1(J_0(N)(\mathbb{C}),\mathbb{Z})$ rather than the submodule H_e used in [1]. So there is no analog of the term $H^+/(I_eH)^++H_e^+$ in our formula. Moreover, the factor c_M (resp. n) in the left hand side of [1][Eq. (1)] does not occur in our formula because we disregard $l=2$ (resp. we do not introduce the Eisenstein ideal into our computation).

It is expected that the only primes l dividing $|J(\mathbb{Q})|$, $|\hat{J}(\mathbb{Q})|$ and $|\Phi|$ are those dividing the numerator n of $(N-1)/12$. Assuming this and $l \nmid n$ we find that $Ш(J/\mathbb{Q})_{l^\infty}$ is of finite projective dimension over $\mathfrak{T}_l$ (and its fitting ideal should be $\mathfrak{b}_l^{-1}$). It would be interesting to look for examples of pairs (N, l) where $\mathfrak{b}_l$ is nontrivial, $\mathfrak{T}_l$ is nonmaximal and $l \nmid n$. In such a case "being of finite projective dimension" would ascertain a nontrivial module theoretic property of $Ш(J/\mathbb{Q})$.

Motives $M(f)(n)$ for f of weight k. In the critical range $1 \leq n \leq k-1$ Conjecture 2 (apart from the vanishing of $H^1_f(M)$) is known and the results of Kato [50] yield upper bounds for the Selmer group similar to the Birch and Swinnerton-Dyer case discussed above. If k is even and the sign of the functional equation of $L(f,s)$ is -1 a limit formula for $L'(f, \frac{k}{2})$ has been proven by Zhang in [81], generalizing the Gross-Zagier formula in the case $k = 2$.

In the noncritical range $n \geq k$ one has results towards Conjecture 2 (the Beilinson conjecture), involving the construction of a subspace of $H^1_f(M)$ whose image under the regulator is related to $L(f,n)$ (see [68] for $k = 2$ and $n \geq 2$. The case $n \geq k > 2$ has been announced in [30] but has not yet appeared in print). For progress towards computation of the étale regulator of the elements constructed in [30] see [37].

5.5. Totally real fields and assorted results. The results of sections 5.1, 5.4 and 5.3 all have partial analogues for a totally real base field F in place of $\mathbb{Q}$. In this section we fix such a field F.

Abelian extensions of F. The main conjecture of Iwasawa theory, proved by Wiles [79], together with the vanishing of the μ-invariant for totally real fields, very recently shown by Barsky [4], yields the following result. For a CM field L we denote by c the unique complex conjugation of L.

THEOREM 5.20. *(Wiles/Barsky) Let L/F be an abelian extension with group G and so that L is a CM-field. For $j < 0$ let M^- be the direct summand of $h^0(\mathrm{Spec}(L))(j)$ cut out by the rational idempotent associated to $(1 + (-1)^j c)/2$ of $\mathbb{Q}[G]$. Then conjecture 3 holds for M^-, $l \neq 2$, and $\mathfrak{A} = \mathbb{Z}[G]/(1 - (-1)^j c)$ (one also needs to assume that the ray class field of F of conductor ℓ is a CM field to satisfy hypothesis H-0 of [4]).*

PROOF. (Sketch) The l-adic L-function interpolating the (critical) values of L-functions of finite order Hecke characters of F is given by an inverse limit of Stickelberger elements (as in [80][Eq. (1)]), similar to the elements θ_{m_0} discussed in section 5.1. However, there is no analogue of cyclotomic units (unless $F/\mathbb{Q}$ is abelian), and the generalization of Theorem 5.2 can only can be proven for $\Delta^\infty \otimes^{\mathbb{L}}_\Lambda \Lambda/(c+1)$. The analogue of the odd part of Lemma 5.4 is [79][Thm. 1.2, Thm. 1.4], and the vanishing of the μ-invariant of θ_{m_0} and hence of P^∞ is proven recently in [4]. The descent arguments are then identical to those given in section 5.1. □

Remark. For $j = 0$ and (odd) characters χ such that $\chi(l) = 1$ there is no analogue of Lemma 5.11 and therefore a descent along the lines indicated above is not possible. With the results of [80][Thm. 1.3] one might be able to deduce some cases of Conjecture 3 for motives $h^0(\mathrm{Spec}\, L)(\chi)$ cut out by rational characters χ of G and $\mathfrak{A}$ a maximal order in $\mathbb{Q}(\chi)$.

Note that the conditions $j < 0$ and $\chi(c) = (-1)^{j+1}$ ensure that $M(\chi)$ is not only critical but that in fact all the six $\mathbb{Q}$-spaces involved in the definition of $\Xi(M)$ vanish. This is the simplest possible situation of the Tamagawa number conjecture: the L-value is an algebraic number (in the coefficient field A).

Hilbert modular forms. Shouwu Zhang has recently generalized the formula of Gross-Zagier from $X_0(N)$ to arbitrary Shimura curves over totally real fields F [82]. One therefore has a generalization of the Euler system of Heegner points from motives $M(f)$ discussed in section 5.4 to motives attached to Hilbert modular forms f of parallel weight 2 with all the ensuing consequences. The work of Zhang probably represents the most significant advance concerning Conjectures 1-2 in recent years. For illustration we quote one result from [82].

THEOREM 5.21. *(Zhang) Let f be a Hilbert modular newform over F, of parallel weight 2 and level N. Assume that either $[F : \mathbb{Q}]$ is odd or that $\mathrm{ord}_v(N) = 1$ for one place of F, and let $A(f)$ be the abelian variety associated to f. Finally assume that $\mathrm{ord}_{s=1} L(f, s) \leq 1$. Then Conjectures 1 and 2 hold for $M = h^1(A(f))(1)$ and $A = E_f$, the field generated by the Hecke eigenvalues of f. Moreover, $\text{Ш}(A(f))$ is finite or, equivalently, Conjecture $\mathbf{Mot}_l$ holds for all l.*

For more details we refer to the papers by Zhang [82], [83] and Tian [73]. We remark that an analogue for Kato's Euler system is still missing over totally real fields (and may be not expected).

Adjoint motives of Hilbert modular forms. The Taylor-Wiles method has been partially generalized to totally real fields [36] and may be expected to give instances of Conjecture 3, analogous to those in section 5.3. To a certain extent this has been worked out in the recent thesis of Dimitrov [32] (without the precise relation to a motivic period, however). In a slightly different direction, using Fujiwara's work for Hilbert modular forms of CM-type, Hida [44] has recently shown many cases of the anticyclotomic main conjecture for CM-fields.

Hecke Characters. Conjectures 1 and 2 are known for critical motives $M(\Psi)$ where Ψ is any Hecke character of any number field [9]. Using the work of Hida [44] mentioned in the previous paragraph it is quite likely that one can also obtain many instances of Conjecture 3 for such motives (if Ψ is a Hecke character of a CM-field). In the noncritical range Conjectures 1 and 2 are known for Tate motives $h^0(\mathrm{Spec}(L))(j)$ over any number field L and $A = \mathbb{Q}$ [12].

We refer to Ramakrishnan's comprehensive survey article [64] for the state of affairs with regard to Conjectures 1 and 2 circa 1989, before Conjecture 3 was formulated.

Part 3. Determinant Functors: Some Algebra

In order to formulate an equivariant Tamagawa Number Conjecture over a not necessarily commutative semisimple algebra A we need to discuss some algebraic preliminairies. Recall that for a commutative ring R and finitely generated projective R-module P, the rank of P is a locally constant integer valued function

$\mathrm{rank}_R(P) \in H^0(\mathrm{Spec}(R), \mathbb{Z})$. The determinant of P is the invertible R-module

$$\det_R(P) := \bigwedge^{\mathrm{rank}_R(P)} P.$$

A short exact sequence

$$0 \to P_1 \to P_2 \to P_3 \to 0$$

induces a (functorial) isomorphism

$$\det_R(P_2) \cong \det_R(P_1) \otimes \det_R(P_3). \tag{5.42}$$

If $P_2 = P_1 \oplus P_3$, and one does not specify which of P_1 and P_3 one views as the submodule and which as the quotient, there is a sign ambiguity in the isomorphism (5.42) (if the tensor product of invertible modules is endowed with its usual commutativity isomorphism $x \otimes y \mapsto y \otimes x$). In order to avoid such ambiguity one has to retain the rank information and view the determinant as a functor

$$\mathrm{Det}_R : (\mathrm{PrMod}(R), is) \to \mathrm{Inv}(R), \quad P \mapsto \mathrm{Det}_R(P) := (\det_R(P), \mathrm{rank}_R(P))$$

from the category of finitely generated projective R-modules $\mathrm{PrMod}(R)$, and isomorphism of such, to the category $\mathrm{Inv}(R)$ of **graded** invertible R-modules, and isomorphisms of such. The monoidal category $\mathrm{Inv}(R)$ has a modified commutativity constraint involving a sign depending on the grading (see [16][2.5] for more details). The functor Det_R can be extended to perfect complexes and isomorphisms of such, and there is an isomorphism (5.42) for short exact sequences of complexes. All computations in the last section are understood to be performed with Det rather than det. For example, the source of the map ϑ_∞ is really the graded module $(\mathbb{R}, 0)$.

We now indicate how a slightly more abstract point of view on the determinant functor leads to its generalization to **non-commutative** rings.

The category $\mathrm{Inv}(R)$ is an example of a so called **Picard category**:

- All morphisms are isomorphism.
- There is a bifunctor $(L, M) \mapsto L \boxtimes M$ with unit object $\mathbf{1}$, inverses, associativity and commutativity constraint.

Definition. Let R be any ring. A **determinant functor** is a Picard category $\mathcal{P}$, a functor $D : (\mathrm{PrMod}(R), is) \to \mathcal{P}$, and functorial isomorphisms $D(P_2) \cong D(P_1) \boxtimes D(P_3)$ for short exact sequences, satisfying conditions as indicated in [16][2.3].

Theorem 5.22. *(Deligne, [28]) For any ring R there is a universal determinant functor*

$$D_R : (\mathit{PrMod}(R), is) \to V(R).$$

$V(R)$ is called the category of virtual objects of R.

This is a categorical version of the Grothendieck group $K_0(R)$. Indeed, any Picard category $\mathcal{P}$ gives rise to two abelian groups $\pi_0(\mathcal{P})$ and $\pi_1(\mathcal{P})$, the group of isomorphism classes of objects of $\mathcal{P}$ with product induced by $\boxtimes$, and the automorphism group of the unit object $\mathbf{1}$ (or any other object). Deligne also shows that D_R induces isomorphisms

$$K_0(R) \xrightarrow{\sim} \pi_0(V(R)) := \begin{cases} \text{Isomorphism classes of objects} \\ \text{with product induced by } \boxtimes \end{cases}$$
$$K_1(R) \xrightarrow{\sim} \pi_1(V(R)) := \mathrm{Aut}_{V(R)}(\mathbf{1})$$

If R is commutative we have a (monoidal) functor $V(R) \to \mathrm{Inv}(R)$ by universality. This functor induces (split) surjections

$$K_0(R) \twoheadrightarrow \mathrm{Pic}(R) \oplus H^0(\mathrm{Spec}(R), \mathbb{Z}) = \pi_0(\mathrm{Inv}(R))$$
$$K_1(R) \twoheadrightarrow R^\times = \pi_1(\mathrm{Inv}(R))$$

and $V(R) \to \mathrm{Inv}(R)$ is an equivalence of categories if and only if both maps are isomorphisms. For a general commutative ring R this is rarely the case but it is true if R is a product of local rings. Examples are the rings $R = A, A_l := A_{\mathbb{Q}_l}, \mathfrak{A}_l, A_{\mathbb{R}}$ considered in the previous section.

6. Noncommutative Coefficients

In order to formulate Conjectures 1-3 in the commutative case it was necessary to take determinants over $R = A, A_l, \mathfrak{A}_l, A_{\mathbb{R}}$, and for these rings we have an equivalence of categories $V(R) \cong \mathrm{Inv}(R)$. This suggests that the categories $V(R)$ can be used to generalize Conjectures 1-3 to motives M with an action of an arbitrary (semisimple) algebra A, and $\mathbb{Z}$-orders $\mathfrak{A} \subseteq A$ so that there exists a projective, $G_{\mathbb{Q}}$-stable $\mathfrak{A}_l$-lattice $T_l \subseteq M_l$. This is indeed the case. More specifically:

- $\Xi({}_AM)$ is an object of $V(A)$.
- ${}_A\vartheta_\infty : \mathbf{1}_{V(A_{\mathbb{R}})} \cong \Xi({}_AM) \otimes_A A_{\mathbb{R}}$ is an isomorphism in $V(A_{\mathbb{R}})$. Here the tensor product $V \otimes_R R'$ for V an object of $V(R)$ and $R \to R'$ any ring extension has to be understood as the monoidal functor
$$- \otimes_R R' : V(R) \to V(R')$$
induced by the exact functor
$$- \otimes_R R' : \mathrm{PrMod}(R) \to \mathrm{PrMod}(R')$$
and the universal property of $V(R)$.
- $D_{\mathfrak{A}_l}(R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l))$ is an object of $V(\mathfrak{A}_l)$.
- ${}_A\vartheta_l : \Xi(M) \otimes_A A_l \cong D_{\mathfrak{A}_l}(R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)) \otimes_{\mathfrak{A}_l} A_l$ is an isomorphism in $V(A_l)$.

An A-equivariant L-function $L({}_AM, s)$ can be defined as a meromorphic function with values in the center $\zeta(A_{\mathbb{C}})$ of $A_{\mathbb{C}}$ [16][Sec. 4]. Its vanishing order can be viewed as a locally constant function

$$r({}_AM) \in H^0(\mathrm{Spec}(\zeta(A_{\mathbb{R}}), \mathbb{Z})$$

and for any finitely generated A-module P we define $\dim_A P \in H^0(\mathrm{Spec}(\zeta(A), \mathbb{Z})$ as its reduced rank ([16][2.6]).

Conjecture 1 (Final Version):

$$r({}_AM) = \dim_A H^1_f(M^*(1)) - \dim_A H^0_f(M^*(1))$$

The leading coefficient $L^*({}_AM)$ of $L({}_AM,s)$ is a unit in $\zeta(A_{\mathbb{R}})$. There is a reduced norm map

$$\mathrm{Aut}_{V(A_{\mathbb{R}})}(\mathbf{1}) = K_1(A_{\mathbb{R}}) \xrightarrow{\mathrm{nr}} \zeta(A_{\mathbb{R}})^\times$$

which is injective but not necessarily surjective (in the case where $A_{\mathbb{R}}$ has quaternionic Wedderburn components the cokernel of nr is a group of exponent 2). One may pick $\mu \in \zeta(A)^\times$ so that

$$\mu L^*({}_AM) = \mathrm{nr}(L_{\mu,\mathbb{R}})$$

for a (unique) $L_{\mu,\mathbb{R}} \in K_1(A_{\mathbb{R}})$. Then the following conjecture is independent of the choice of μ.

Conjecture 2 (Final Version): The composite morphism

$$\mathbf{1}_{V(A_{\mathbb{R}})} \xrightarrow{L_{\mu,\mathbb{R}}^{-1}} \mathbf{1}_{V(A_{\mathbb{R}})} \xrightarrow{{}_A\vartheta_\infty} \Xi({}_AM) \otimes_A A_{\mathbb{R}}$$

in $V(A_{\mathbb{R}})$ is the image of a morphism $\mathbf{1}_{V(A)} \xrightarrow{L_\mu^{-1}} \Xi({}_AM)$ in $V(A)$ under the scalar extension functor $-\otimes_A A_{\mathbb{R}}$.

Now recall that for any prime number l the reduced norm map $\mathrm{nr}_l : K_1(A_l) \to \zeta(A_l)^\times$ is an isomorphism so that there is a unique element $\mu_l \in K_1(A_l)$ with $\mathrm{nr}_l(\mu_l) = \mu$. Assuming Conjecture 2, the morphism

$$L_l^{-1} : \mathbf{1}_{V(A_l)} \xrightarrow{\mu_l} \mathbf{1}_{V(A_l)} \xrightarrow{L_\mu^{-1}\otimes_A A_l} \Xi({}_AM) \otimes_A A_l$$

is independent of the choice of μ.

Conjecture 3 (Final Version): The composite morphism

$$\mathbf{1}_{V(A_l)} \xrightarrow{L_l^{-1}} \Xi({}_AM) \otimes_A A_l \xrightarrow{{}_A\vartheta_l} D_{\mathfrak{A}_l}(R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)) \otimes_{\mathfrak{A}_l} A_l$$

in $V(A_l)$ is the image of a morphism $\mathbf{1}_{V(\mathfrak{A}_l)} \to D_{\mathfrak{A}_l}(R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l))$ in $V(\mathfrak{A}_l)$ under the scalar extension functor $-\otimes_{\mathfrak{A}_l} A_l$.

This conjecture is only of interest for non-maximal orders $\mathfrak{A}$, in the sense that it is implied by Conjecture 3 for *commutative* coefficients (for certain motives related to M) if $\mathfrak{A}$ is maximal [16][Prop. 4.2].

A Reformulation. Let $V(\mathfrak{A}_l, \mathbb{Q}_l)$ denote the Picard category whose objects are pairs (V, τ) where V is an object of $V(\mathfrak{A}_l)$ and $\tau : V \otimes_{\mathfrak{A}_l} A_l \cong \mathbf{1}_{V(A_l)}$ is an isomorphism. Then $\pi_0(V(\mathfrak{A}_l, \mathbb{Q}_l))$ is the usual relative $K_0(\mathfrak{A}_l, \mathbb{Q}_l)$ [16][Prop. 2.5] and the object $D_{\mathfrak{A}_l}(R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l))$, together with the map in Conjecture 3 defines a class $T\Omega(M, \mathfrak{A})_l \in K_0(\mathfrak{A}_l, \mathbb{Q}_l)$. Conjecture 3 is equivalent to the vanishing

$$T\Omega(M, \mathfrak{A})_l = 0$$

in $K_0(\mathfrak{A}_l, \mathbb{Q}_l)$. Moreover, as shown in [16][Sec. 3], one can define a class $T\Omega(M, \mathfrak{A}) \in K_0(\mathfrak{A}, \mathbb{R})$ so that Conjecture 2 holds if and only if

$$T\Omega(M, \mathfrak{A}) \in K_0(\mathfrak{A}, \mathbb{Q}) \subseteq K_0(\mathfrak{A}, \mathbb{R}).$$

If this is the case then $T\Omega(M, \mathfrak{A})_l$ coincides with the l-component of $T\Omega(M, \mathfrak{A})$ under the decomposition $K_0(\mathfrak{A}, \mathbb{Q}) \cong \bigoplus_l K_0(\mathfrak{A}_l, \mathbb{Q}_l)$.

7. The Stark Conjectures

There is only one class of examples where Conjecture 3 with noncommutative coefficients has been considered so far, essentially those of our Example a) with $j \leq 0$. Traditionally the motivation in this work is to investigate the $\mathbb{Z}[G]$-structure of the unit group $\mathcal{O}_L^\times$ (or of $K_{1-2j}(\mathcal{O}_L)$ for $j < 0$ [14], [23]).

The following theorem indicates how previous work is implied by Conjecture 3.

THEOREM 7.1. *Let L/K be a Galois extension of number fields with group G and put $M = h^0(\mathrm{Spec}(L))$, $A = \mathbb{Q}[G]$. Denote by $\mathfrak{M}$ a maximal order of A containing $\mathbb{Z}[G]$.*

a) *Conjecture 1 is true.*
b) *Conjecture 2 is equivalent to Stark's Conjecture as given in [72][Ch. I, Conj. 5.1].*
c) *Conjecture 3 for $\mathfrak{A} = \mathfrak{M}$ (and all l) is equivalent to the strong Stark Conjecture as formulated by Chinburg [20][Conj. 2.2].*
d) *Conjecture 3 for $\mathfrak{A} = \mathbb{Z}[G]$ (and all l) implies Chinburg's conjecture [21]*

$$\omega(L/K) + \Omega(L/K, 3) = 0.$$

PROOF. We refer to [17] for details of this theorem. The key fact is the existence of a perfect complex of $\mathbb{Z}[G]$-modules Ψ which underlies both $R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)$ and $\Xi({}_AM)$, and which is quasi-isomorphic to a complex used in Chinburg's work (a so called Tate-sequence). The proof of this quasi-isomorphism is rather involved and given in [15]. The implication in d) follows because the image of $T\Omega(M, \mathfrak{A})$ under the natural map $K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}[G])$ is $\omega(L/K) + \Omega(L/K, 3)$. □

The supply of known cases of Conjecture 3 for $M = h^0(\mathrm{Spec}(L))$ and noncommutative $\mathfrak{A} = \mathbb{Z}[G]$ is rather scarce.

THEOREM 7.2. *(Burns-Flach/Chinburg) Let l be any rational prime such that $l \equiv 1 \mod 12$ and also $\left(\frac{l}{7}\right) = -\left(\frac{l}{5}\right) = 1$. There is a unique extension $L/\mathbb{Q}$ with Galois group the quaternion group Q_8 of order 8, ramified precisely at $\{3, 5, 7, l, \infty\}$ and containing $\mathbb{Q}(\sqrt{21}, \sqrt{5})$ [22][Prop. 4.1.3]. Then Conjecture 3 holds for $M = h^0(\mathrm{Spec}(L))$ and $\mathfrak{A} = \mathbb{Z}[Q_8]$.*

PROOF. Put $V = \mathrm{Gal}(\mathbb{Q}(\sqrt{21}, \sqrt{5})/\mathbb{Q})$. One verifies that the kernel of the homomorphism

$$K_0(\mathbb{Z}[Q_8], \mathbb{R}) \to K_0(\mathfrak{M}, \mathbb{R}) \times K_0(\mathbb{Z}[Q_8]) \times K_0(\mathbb{Z}[V], \mathbb{R})$$

given by the natural maps to each factor, is trivial (although the kernel of each map to a pair of factors is not) [17][Lemma 4]. The image of $T\Omega(M, \mathbb{Z}[Q_8])$ in $K_0(\mathfrak{M}, \mathbb{R})$ is trivial because of Theorem 7.1 c) and the fact that the Strong Stark Conjecture is known for groups all of whose characters are rational (such as Q_8) [72][Ch.II, Th. 6.8]. The image of $T\Omega(M, \mathbb{Z}[Q_8])$ in $K_0(\mathbb{Z}[Q_8])$ is trivial because of Theorem 7.1 d) and computations of Chinburg showing the vanishing of $\omega(L/\mathbb{Q}) + \Omega(L/\mathbb{Q}, 3)$ [22]. Finally the image in $K_0(\mathbb{Z}[V], \mathbb{R})$ is trivial by functoriality [16][Prop. 4.1b)] and Theorem 5.1. For a nice concrete interpretation of this last vanishing see [52]. □

One may ask whether there is hope of verifying Conjecture 3, or even just Conjecture 2, in genuine non-abelian cases, for example where $G = \mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the alternating group A_5. This group has five irreducible characters $\chi_1, \chi_3, \bar{\chi}_3, \chi_4, \chi_5$ where χ_3 takes values in $\mathbb{Q}(\sqrt{5})$ and the other characters are rational. For rational characters χ the strong Stark conjecture for $L^*(\chi, 0)$ is known by [72][Ch.II, Th. 6.8]. For χ_3 one is in the favorable situation where $\mathrm{ord}_{s=0} L(\chi_3, s) = 1$ (if L is imaginary) and one might be able to verify the Stark conjecture numerically (see [67] for an example of such verifications).

Conjecture 3 for $\mathfrak{A} = \mathbb{Z}[G]$, on the other hand cannot be checked character by character (but rather entails some relationship among all $L^*(\chi_i, 0)$). A numerical verification would possibly involve a computation of the full group of units of L with its $\mathbb{Z}[G]$-structure, and seems currently out of reach. We remark that Bley [11] has verified Conjecture 3 numerically for certain cyclic extensions $L/\mathbb{Q}(\sqrt{5})$ of order 3 and 5.

Beyond numerical verifications, one has to prove limit formulas such as the one in section 5.1 relating Dirichlet L-series and cyclotomic units. There is now a large supply of extensions $L/\mathbb{Q}$ with group A_5 where it is known that $L(\chi_i, s)$ coincides with the L-function attached to an automorphic form on $GL_i(\mathbb{Q})$ for all i (combine [19] with arguments as in [65] and the recent preprint [51]). This does not seem to help in proving any relationship between the leading coefficient $L^*(\chi_i, 0)$ and units in L, however.

For more information on the Stark conjectures we refer to the article of David Burns in this volume.

APPENDIX: On the vanishing of μ-invariants

by C. Greither

The purpose of this appendix is to give a proof that the μ-invariant of an often-used Iwasawa module (the one referred to as "limit of units modulo limit of cyclotomic units") is zero for the cyclotomic p-tower over every absolutely abelian field K and for every prime p. The stress lies on the two occurrences of "every": for odd p this seems to be well-known, and for $p = 2$ and K a full cyclotomic field, the result may be extracted from work of Kuz'min, see below. So if there is anything really new here, it is the case $p = 2$ and K not a full cyclotomic field. But for the reader's convenience, and since it does not cost anything extra, we shall give a unified argument for all p and K, with the only restriction that K is assumed imaginary; we will provide a few comments on the real case at the end. The author would like to use the opportunity to point out the following: In the author's paper [41], Theorem 3.1 and the subsequent remark c) claim an equality of characteristic ideals, which would imply the desired vanishing of the μ-invariant. But this implication does not stand since the argument in loc. cit. only yields that the characteristic ideals are equal up to a power of p. Thanks to Annette Huber and Matthias Flach, who (independently) noticed this.

Let us define our objects precisely before we state the result. Let p be a prime number (we repeat that $p = 2$ is permitted), fixed in the sequel. Let us also fix an absolutely abelian field K with Galois group G over the rationals. Let K_∞ be the cyclotomic $\mathbb{Z}_p$-extension of K; write Γ for $\mathrm{Gal}(K_\infty/K)$ and Γ_n for $\mathrm{Gal}(K_\infty/K_n)$ where K_n is the n-th layer of K_∞. We subsume p-adification in our notation by writing E_n for $\mathbb{Z}_p \otimes \mathcal{O}^*_{K_n}$ and C_n for $\mathbb{Z}_p \otimes \mathrm{Sinn}_{K_n}$ where Sinn_L denotes the group of circular units in Sinnott's sense, for any abelian field L. Let E (respectively C) denote the limit of the projective system $(E_n)_n$ resp. $(C_n)_n$ via the norm maps. (These are often written E_∞ and C_∞.) Finally we put $\Lambda = \mathbb{Z}_p[[\Gamma]]$ as customary. The result reads:

THEOREM. *Under the above assumption that K is absolutely abelian and imaginary, and with the notation just introduced, the Λ-module E/C has μ-invariant zero.*

Apart from a few short remarks at the end, the rest of this appendix is occupied by the proof of this result. The essential input comes from Sinnott's work: from his main theorem [70][p.182] we immediately deduce the existence of a constant c_K depending only on K (not on n) such that

$$[E_n : C_n] = c_K \cdot h(K_n^+)$$

for all sufficiently large $n \in \mathbb{N}$. This together with the Ferrero-Washington theorem implies that there exists another constant $\lambda = \lambda_K$ such that the index $[E_n : C_n]$ is $O(p^{\lambda n})$ as a function of n. This does not *directly* imply the nullity of $\mu(E/C)$, since there certainly exists a projective system $(X_n)_n$ of $\mathbb{Z}_p[\Gamma/\Gamma_n]$-modules with quite slow growth and such that the limit is $\Lambda/p\Lambda$: take $X_n = \Lambda/(p, T^n)$. The point is that the "natural" system $(E_n/C_n)_n$ cannot misbehave in that way, as we will show.

We will use an ad hoc notion. A family (not necessarily a projective system) $(A_n)_n$ of finite $\mathbb{Z}_p$-modules is called *tame* if the following holds: There exist positive integers c_1 and c_2 and submodules $A'_n \subset A_n$ requiring at most c_2 $\mathbb{Z}_p$-generators each, such that the indices $[A_n : A'_n]$ are $O(p^{c_1 n})$ as a function of n. We observe: Bounded families are tame; and if $U_n \subset A_n$ for all n, then the family $(A_n)_n$ is tame iff both families $(U_n)_n$ and $(A_n/U_n)_n$ are tame.

These observations easily lead to the following consequence: If the Λ-module X has $\mu(X) > 0$, then the family of Γ_n-coinvariants $(X_{\Gamma_n})_n$ will not be tame. We therefore just need to show that the family $\big((E/C)_{\Gamma_n}\big)_n$ is tame.

Let i_n be the canonical map $E_{\Gamma_n} \to E_n$ and let B_n stand for its kernel. Write B'_n for the image of B_n in $(E/C)_{\Gamma_n}$. Then the induced map $j_n : \big((E/C)_{\Gamma_n}\big)/B'_n \to E_n/C'_n$ is injective, where C'_n is the image of the n-th projection $C \to C_n$. Alternatively, C'_n can be described as the subgroup of stable norms inside C_n, more explicitly: C'_n is the set of all $x \in C_n$, which are in the image of $C_m \to C_n$ for all $m \geq n$.

Assume we can establish the following two facts:

(1) $(E_n/C'_n)_n$ is tame; and

(2) $(B'_n)_n$ is tame.

Then from the injectivity of j_n and the abovementioned simple properties of tame families we get tameness of $(E/C)_{\Gamma_n}$, and we will be done. Fact (1) is, obviously, a consequence of the tameness of $(E_n/C_n)_n$ (which we know to hold) and another fact:

(3) $(C_n/C'_n)_n$ is tame.

It thus remains to establish (2) and (3); in fact we will show tameness with the parameter c_1 set to zero, that is, the number of required $\mathbb{Z}_p$-generators is bounded in both families.

Proof of Fact (3): One has to recall the construction of Sinnott units. Consider the following group D_n of p-adified Sinnott circular numbers: D_n is generated over $\mathbb{Z}_p[G]$ by all elements $z_d = \mathrm{N}_{\mathbb{Q}(d)/\mathbb{Q}(d)\cap K_n}(1 - \zeta_d)$, where $\mathbb{Q}(d)$ is short for $\mathbb{Q}(\zeta_d)$ and $d \neq 1$ divides the conductor f_n of K_n and $(d, f_n/d) = 1$. Then $C_n = D_n \cap E_n$, by [57][Prop. 1] and an easy extra argument to eliminate those d which are not coprime to f_n/d. (For $p = 2$, one also has to put -1 into D_n.) Since a Sinnott circular number as above can only be a non-unit at places dividing the conductor of K_n, and the valuations at all places above the same rational place are the same, the intersection of the stable norms D'_n with C_n gives the stable norms C'_n. It therefore suffices to show tameness of $(D_n/D'_n)_n$.

There is some n_0 such the conductor of K_n is exactly divisible by p^{n+n_0+1} for all large n. Look at divisors d of f_n which are divisible by p, and note that this forces d to be exactly divisible by p^{n+n_0+1} since we only consider d which are coprime to f_n/d. Using Galois theory one can check that for n large enough, the degree of the field $\mathbb{Q}(pd) \cap K_{n+1}$ over $\mathbb{Q}$ is exactly p times the degree of $\mathbb{Q}(d) \cap K_n$. If n is large enough to make this happen, then the obvious fact that $1 - \zeta_d$ is the norm of $1-\zeta_{pd}$ in the degree p extension $\mathbb{Q}(pd)/\mathbb{Q}(d)$, implies by chasing a diagram of fields that z_d is the norm of z_{pd} in K_{n+1}/K_n.

Therefore for any large n, all z_d with d divisible by p are stable norms. There remain only those z_d with d prime to p, so d must divide $\mathrm{cond}(K)$, so z_d is actually in K, and a unit outside $\mathrm{cond}(K)$. This gives an a priori bound of the number of $\mathbb{Z}_p$-generators required for the quotient D_n/D'_n, and Fact (3) is proved.

Proof of Fact (2): We will find a bound c_2 so that all B_n are c_2-generated over $\mathbb{Z}_p$. One easily sees that $B_n = \operatorname{proj\,lim}_{m\geq n} \mathrm{H}^1(\Gamma_{m,n}, E_m)$, with $\Gamma_{m,n} = \mathrm{Gal}(K_m/K_n)$. If E'_m denotes $\mathbb{Z}_p \otimes \mathcal{O}_{K_m}[1/p]^*$ (the p-adified p-units in K_m), then we have short exact sequences

$$1 \to E_m \to E'_m \to V_m \to 0,$$

where V_m is defined by the sequence. Then the $\mathbb{Z}_p$-rank of the free $\mathbb{Z}_p$-module V_m equals the number of p-adic places of K_m and is therefore bounded as $m \to \infty$, by c' say. Moreover the induced maps $V_{m+1} \to V_m$ are all monic, so $V = \operatorname{proj\,lim} V_m$ is a free $\mathbb{Z}_p$-module of rank at most c' as well.

By an old result of Iwasawa [47], the orders of $\mathrm{H}^1(\Gamma_{m,n}, E'_m)$ are bounded independently of m and n, and in particular all these groups are c''-generated for some c''. Looking at the cohomology sequence coming from the short exact sequence above we therefore see that we will be done, with $c_2 = c' + c''$, as soon as we can show that the modules $\operatorname{proj\,lim}_{m\geq n} \mathrm{H}^0(\Gamma_{m,n}, V_m)$ are d'-generated over $\mathbb{Z}_p$ for all large n. But this is clear since $\operatorname{proj\,lim}_{m\geq n} \mathrm{H}^0(\Gamma_{m,n}, V_m) \cong \mathrm{H}^0(\Gamma_n, V)$, and V is c'-generated.

This finishes the proof of Fact (2), and the proof of the theorem.

Remarks: (a) The case $p = 2$ and K a full cyclotomic field can be deduced from a result of Kuz'min [56, Thm. 3.1]; a little argument is necessary for which we refer to [62, p.77].

(b) One can show the following (we do not give the proofs, which use the same methods). If one takes K a real abelian field, and retains the definition of C_n and C as above, the result is no longer true for $p = 2$: instead one gets $\mu(E/C) = [K : \mathbb{Q}]$. If one replaces Sinnott units by modified Sinnott units, i.e. by the groups denoted $C_{1,n}$ by Sinnott [70, p.182], and defines $\tilde{C} = \operatorname{proj\,lim}(\mathbb{Z}_2 \otimes C_{1,n})$, then $\mu(E/\tilde{C})$ turns out to be zero. One can even show that $\tilde{C}/C$ is isomorphic to $\Lambda/(2)^{[K:\mathbb{Q}]}$. By definition, $C_{1,n}$ is the group of all units of K_n whose squares are Sinnott units, and the main point is that in a precise sense, almost all Sinnott units are squares for real fields K.

(c) If K is of odd degree over $\mathbb{Q}$, then $\mu(E/\tilde{C}) = 0$ already essentially follows from work of Gillard [39] and, again, the theorem of Ferrero-Washington.

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