Contemporary Mathematics

Quasi-invariant measures for continuous group actions

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Dedicated to Simon Thomas on his 60th birthday

ABSTRACT. The class of ergodic, invariant probability Borel measure for the shift action of a countable group is a G_δ set in the compact, metrizable space of probability Borel measures. We study in this paper the descriptive complexity of the class of ergodic, quasi-invariant probability Borel measures and show that for any infinite countable group Γ it is $\mathbf{\Pi^0_3}$-hard, for the group $\mathbb{Z}$ it is $\mathbf{\Pi^0_3}$-complete, while for the free group $\mathbb{F}_\infty$ with infinite, countably many generators it is $\mathbf{\Pi^0_\alpha}$-complete, for some ordinal α with $3 \leq \alpha \leq \omega + 2$. The exact value of this ordinal is unknown.

1. Introduction

For any Polish space X, let $P(X)$ be the Polish space of probability Borel measures on X with the usual topology (see, e.g., [**K**, 17.E]). It is compact, metrizable, if X is compact, metrizable. Any $f\colon X \to Y$ induces the map $f_*\colon P(X) \to P(Y)$, defined by $f_*(\mu)(B) = \mu(f^{-1}(B))$.

If E is an equivalence relation on X, a measure $\mu \in P(X)$ is **ergodic** for E if for any Borel E-invariant set $A \subseteq X$, $\mu(A) \in \{0, 1\}$. We denote by ERG_E the set of such measures. Similarly if $a\colon \Gamma \times X \to X$ is an action of a group Γ on X, a measure μ is ergodic for a if for any invariant under a Borel set A, we have $\mu(A) \in \{0, 1\}$. We denote again

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by ERG_a the set of such measures. Clearly $\mathrm{ERG}_a = \mathrm{ERG}_{E_a}$, where E_a is the equivalence relation induced by (the orbits of) the action a.

Consider now a continuous action a of a countable (discrete) group Γ on a compact, metrizable space K. If a is understood from the context, we write $\gamma \cdot x$ instead of $a(\gamma, x)$. We also let $\gamma^a(x) = a(\gamma, x)$. It is a standard fact that the set INV_a of invariant measures for a is closed in $P(K)$ and the set EINV_a of invariant, ergodic measures for a is G_δ in $P(K)$ (see, e.g., [**G**, Theorem 4.2]).

Recall that $\mu \in P(K)$ is called **quasi-invariant** for the action a if for any $\gamma \in \Gamma$, $\gamma \cdot \mu \sim \mu$, where $\sim$ denotes measure equivalence and $\gamma \cdot \mu = (\gamma^a)_*(\mu)$. Denote by QINV_a the set of quasi-invariant measures for a and by EQINV_a the subset of ergodic, quasi-invariant measures for a. Since the relation $\sim$ of measure equivalence is $\mathbf{\Pi}^0_3$ in $P(K)^2$, it follows that QINV_a is $\mathbf{\Pi}^0_3$ in $P(K)$. From a (more general) result of Ditzen in [**D**], it follows that ERG_a is Borel and, again from a (more general) result in Louveau-Mokobodzki [**LM**, page 4823], this can be improved to $\mathrm{ERG}_a \in \mathbf{\Pi}^0_{\omega+2}$. Thus $\mathrm{EQINV}_a = \mathrm{ERG}_a \cap \mathrm{QINV}_a$ is also $\mathbf{\Pi}^0_{\omega+2}$ in $P(K)$.

In this paper we are interested in the Borel complexity of the sets QINV_a and EQINV_a. To avoid technical complications involving the topology of K, we will consider here the case where K is 0-dimensional and thus can be viewed as a closed subspace of the Cantor space $\mathcal{C} = 2^{\mathbb{N}}$. Under these circumstances, the action a of Γ on K can be topologically embedded, via the map $f(x) = (\gamma^{-1} \cdot x)_\gamma$, into the shift action s_Γ of Γ on $\mathcal{C}^\Gamma$. Therefore QINV_a and EQINV_a are Wadge reducible, via the continuous map $\mu \mapsto f_*(\mu)$, to QINV_{s_Γ} and $\mathrm{EQINV}_{s_\Gamma}$, resp. Recall that if $A \subseteq X, B \subseteq Y$, then A is **Wadge reducible** to B if there is a continuous function $f\colon X \to Y$ such that $A = f^{-1}(B)$. In this case we put $A \leq_W B$. We will thus focus our attention on the study of the Borel complexity of the quasi-invariant and ergodic, quasi-invariant measures for the shift action. For convenience we write

$$\mathrm{QINV}_\Gamma = \mathrm{QINV}_{s_\Gamma}, \mathrm{ERG}_\Gamma = \mathrm{ERG}_{s_\Gamma}, \mathrm{EQINV}_\Gamma = \mathrm{EQINV}_{s_\Gamma}.$$

We prove below the following results, where for a class Φ of sets in Polish spaces, a set $A \subseteq X$, X a Polish space, is called **Φ-hard** if for any $B \subseteq Y$, Y a 0-dimensional Polish space, with $B \in \Phi$, we have $B \leq_W A$. If in addition $A \in \Phi$, then A is called **Φ-complete**.

THEOREM 1. *For any infinite, countable group* Γ, QINV_Γ *is* $\mathbf{\Pi}^0_3$-*complete and* $\mathrm{ERG}_\Gamma, \mathrm{EQINV}_\Gamma$ *are* $\mathbf{\Pi}^0_3$-*hard.*

THEOREM 2. *The set* $\mathrm{EQINV}_{\mathbb{Z}}$ *is* $\mathbf{\Pi}^0_3$-*complete.*

THEOREM 3. *Let* $\mathbb{F}_\infty$ *be the group with infinite, countably many generators. Then there is a countable ordinal* $\alpha_\infty \geq 3$ *such that the set* $\mathrm{EQINV}_{\mathbb{F}_\infty}$ *is* $\mathbf{\Pi}^0_{\alpha_\infty}$*-complete.*

Thus $3 \leq \alpha_\infty \leq \omega + 2$.

Problem 4. *Calculate* α_∞.

We note that from Theorem 3 it follows that $\mathrm{EQINV}_\Gamma \in \mathbf{\Pi}^0_{\alpha_\infty}$, for any countable group Γ.

Remark 5. The proof of Theorem 3 in Section 4 below also shows that for any countable group Γ that can be mapped onto the direct sum of infinite, countably many copies of itself, there is a countable ordinal α_Γ (thus $3 \leq \alpha_\Gamma \leq \omega + 2$) such that EQINV_Γ is $\mathbf{\Pi}^0_{\alpha_\Gamma}$-complete.

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2. Proof of Theorem 1

We first note the following standard fact.

Lemma 2.1. *For any continuous action* a *of a countable group* Γ *on a compact, metrizable space* K, $\mathrm{ERG}_a \leq_W \mathrm{EQINV}_a$.

PROOF. Let $\Gamma = \{\gamma_n \colon n \in \mathbb{N}\}$. The map $\mu \in P(K) \mapsto \sum_n 2^{-(n+1)}\gamma_n \cdot \mu \in P(K)$ is a continuous reduction of ERG_a to EQINV_a. □

Thus to complete the proof of Theorem 1, it is enough to show that QINV_Γ is $\mathbf{\Pi}^0_3$-complete and that ERG_Γ is $\mathbf{\Pi}^0_3$-hard.

(A) QINV_Γ is $\mathbf{\Pi}^0_3$-complete.

Let X be a perfect Polish space and Γ an infinite, countable group, which acts freely and continuously on X. Put

$$\begin{aligned} S &= \{(x_n) \in X^{\mathbb{N}} \colon \{x_n \colon n \in \mathbb{N}\} \text{ is } \Gamma\text{-invariant}) \\ &= \{(x_n) \in X^{\mathbb{N}} \colon \forall n \forall \gamma \exists m (\gamma \cdot x_n = x_m)\} \end{aligned}$$

Proposition 2.2. S *is not* G_δ.

PROOF. First notice that S is dense: Given $U_0, \dots, U_{k-1}$ non-$\emptyset$ open in X consider $U_0 \times \cdots \times U_{k-1} \times X^{\mathbb{N}}$. We will show that it intersects S. Pick $x_i^0 \in V_i, i < k$. Then clearly there are $x_k^0, x_{k+1}^0, \dots$ such that $(x_n^0) \in S$.

So if S is G_δ, it is comeager. We will show that there is a dense G_δ set G such that $G \cap S = \emptyset$, a contradiction.

Let $\gamma \neq 1, \gamma \in \Gamma$ and put

$$G = \{(x_n) \colon \forall m (\gamma \cdot x_0 \neq x_m)\}.$$

Clearly $G \cap S = \emptyset$. Now

$$G = \bigcap_m G_m, \text{ where}$$

$$G_m = \{(x_n) \colon \gamma \cdot x_0 \neq x_m\}.$$

Clearly G_m is dense, open, so G is comeager. □

Let now K be perfect, compact, metrizable and let a be a free, continuous action of Γ on K.

Proposition 2.3. QINV_a *is not* G_δ *in* $P(K)$.

Proof. It is enough to find a continuous function

$$F \colon K^{\mathbb{N}} \to P(K)$$

such that $F^{-1}(\mathrm{QINV}_a) = S$, where S is as above for (K, Γ).

Put

$$F((x_n)) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \delta_{x_n},$$

where δ_x is the Dirac measure at $x \in K$.

Claim. F is continuous.

Proof. We need to check that if $f \in C(K)$, and $(x^i_n) \to (x_n)$ in $K^{\mathbb{N}}$, then $F((x^i_n))(f) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(x^i_n) \to F((x_n))(f) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(x_n)$, which is clear as $f(x^i_n) \to f(x_n), \forall n$.

Claim. $F^{-1}(\mathrm{QINV}_\Gamma) = S$.

Proof. If $(x_n) \in S$, then clearly $\gamma \cdot (F(x_n)) \sim F((x_n)), \forall \gamma \in \Gamma$. Conversely assume $(x_n) \notin S$. Let then n, γ be such that $\forall m (\gamma \cdot x_n \neq x_m)$. Then $\gamma \cdot F((x_n)) \not\sim F((x_n))$. □

Thus we have shown:

Proposition 2.4. *Let a be a continuous and free action of an infinite countable group Γ on the perfect, compact metrizable space K. Then* QINV_a *is not* G_δ *in* $P(K)$.

Let now $Q = \{x \in \mathcal{C} \colon x(n) = 0 \text{ for all but finitely many } n\}$. Then Q is F_σ in the Cantor space $\mathcal{C}$ and for any Polish space X and Borel set $A \subseteq X$, if A is not G_δ, then $Q \leq_W A$ (see [**K**, 24.20 and 22.13]). Thus we have:

Corollary 2.5. *Let a be a continuous and free action of an infinite, countable group Γ on the perfect, compact metrizable space K Then there is a continuous function $f\colon \mathcal{C} \to P(K)$ with $f^{-1}(\mathrm{QINV}_a) = Q$.*

Consider now the set $Q^{\mathbb{N}} \subseteq \mathcal{C}^{\mathbb{N}}$. It is known that $Q^{\mathbb{N}}$ is a $\mathbf{\Pi}^0_3$-complete set (see [**K**, page 179] where this set is denoted by P_3). Let now $K = \mathcal{C}^{\Gamma}$ with the shift action. By a result of Gao-Jackson-Seward [**GJS**, 3.7], there are infinitely many, pairwise disjoint, invariant compact subsets K_n of K on which Γ acts minimally and freely. Note that each K_n is perfect. By the preceding corollary, there is a continuous function $f_n\colon \mathcal{C} \to P(K_n)$ such that $f_n^{-1}(\mathrm{QINV}_{a_n}) = Q$, where a_n is the restriction of the shift action to K_n. Define now $f\colon \mathcal{C}^{\mathbb{N}} \to P(K)$ by $f((x_n)) = \sum_n \frac{1}{2^{n+1}} f_n(x_n)$. Then f is continuous and $f^{-1}(\mathrm{QINV}_\Gamma) = Q^{\mathbb{N}}$, so QINV_Γ is $\mathbf{\Pi}^0_3$-complete.

(B) ERG_Γ is $\mathbf{\Pi}^0_3$-hard.

This follows from the following more general result, where a Borel equivalence relation E on a Polish space X is **smooth** if there is a Borel map $f\colon X \to Y$, Y a Polish space, such that $xEy \iff f(x) = f(y)$.

Theorem 2.6. *Let E be a non-smooth, Borel equivalence relation on a Polish space X. Then ERG_E is $\mathbf{\Pi}^0_3$-hard.*

Proof. Let E_0^k be the equivalence relation of $k^{\mathbb{N}}$ given by

$$(x_n) E_0^k (y_n) \iff \exists n \forall m \geq n (x_m = y_m).$$

Then by [**HKL**], E_0^3 can be continuously embedded, say by the function $f\colon 3^{\mathbb{N}} \to X$, into E. The function f_* from $P(3^{\mathbb{N}})$ to $P(X)$ is continuous and μ is ergodic for E_0^3 iff $f_*(\mu)$ is ergodic for E. It is thus enough to prove this result for $E = E_0^3$.

Consider the $\mathbf{\Pi}^0_3$-complete set $P_3 = Q^{\mathbb{N}} \subseteq \mathcal{C}^{\mathbb{N}}$, as in the paragraph following Corollary 2.5. We will define a continuous function $f\colon \mathcal{C}^{\mathbb{N}} \times \mathcal{C} \to 3^{\mathbb{N}}$ as follows:

Fix a bijection $\langle \cdot, \cdot \rangle \colon \mathbb{N}^2 \to \mathbb{N}$. Define first a function $\bar{f}$ by:

$$\bar{f}((a_k), x)(\langle n, m \rangle) = x(n+m), \text{if } a_n(m) = 0; x|[n, n+m], \text{if } a_n(m) = 1.$$

Let then $f((a_k), x) = y$, where letting $y_n(m) = y(\langle n, m \rangle)$, y_n is equal to:

$$2\ \bar{f}((a_k), x)(\langle n, 0 \rangle)\ 2\ \bar{f}((a_k), x)(\langle n, 1 \rangle)\ 2\ \cdots 2\ \bar{f}((a_k), x)(\langle n, m \rangle)\ 2\ \cdots,$$

which is the concatenation of 2 followed by $\bar{f}((a_k), x)(\langle n, 0 \rangle)$ followed by 2 followed by $\bar{f}((a_k), x)(\langle n, 1 \rangle)$ ...

Since $y(\langle n, m \rangle)$ depends only on $a_n(l)$, for $l \leq m$, and $x(n), \ldots, x(n+m)$, it is clear that f is continuous.

It is also clear that for each fixed (a_k), the section $f_{(a_k)}(x) = f((a_k), x)$ is injective. For each $(a_k) \in \mathcal{C}^{\mathbb{N}}$ let now

$$\mu((a_k)) = (f_{(a_k)})_*(\lambda),$$

where λ is the usual product measure on $\mathcal{C}$. The function $\mu\colon \mathcal{C}^{\mathbb{N}} \to P(3^{\mathbb{N}})$ is continuous, so its is enough to show that

$$(a_k) \in P_3 \iff \mu((a_k)) \in \mathrm{ERG}_{E_0^3}.$$

(A) Let $(a_k) \in P_3$. We claim then that $xE_0^2y \implies f_{(a_k)}(x)E_0^3 f_{(a_k)}(y)$. Indeed, if xE_0y, say $x(k) = y(k)$ for all $k \geq n_0$, then for $n \geq n_0$, clearly $f_{(a_k)}(x)(\langle n, m\rangle) = f_{(a_k)}(y)(\langle n, m\rangle), \forall m$. Let also m be large enough so that $a_n(m) = 0$, for all $n < n_0$ and all $m \geq m_0$. Then for some k_0 and all $m \geq k_0, n < n_0$, we have $f_{(a_k)}(x)(\langle n, m\rangle) = f_{(a_k)}(y)(\langle n, m\rangle)$, so $f_{(a_k)}(x)E_0^3 f_{(a_k)}(y)$.

Thus if $A \subseteq 3^{\mathbb{N}}$ is Borel, E_0^3-invariant, then $f^{-1}_{(a_k)}(A)$ is Borel E_0^2-invariant, so, since λ is ergodic for E_0^2, it has λ-measure 0 or 1, and thus A has $\mu((a_k))$-measure 0 or 1. So $\mu((a_k)) \in \mathrm{ERG}_{E_0^3}$.

(B) Let $(a_k) \notin P_3$. Fix then n_0 and $1 < m_0 < m_1 < m_2 \dots$ be such that $a_{n_0}(m_i) = 1, \forall i$. Fix also a tree $T \subseteq 2^{<\mathbb{N}}$ such that $0 < \lambda([T]) < 1$. Put

$$B = \bigcup_{s \in 2^{n_0}} N_s \star [T],$$

where for $s \in 2^{n_0}$:

$$N_s \star [T] = \{a \in \mathcal{C}\colon s \subseteq a \ \& \ (a_{n_0}, a_{n_0+1}, \dots) \in [T]\}.$$

Then $\lambda(B) = \lambda([T]) \in (0,1)$. Put $f_{(a_k)}(B) = C$ and $A = [C]_{E_0^3}$. Then A is Borel, E_0^3-invariant and we will show that $f^{-1}_{(a_k)}(A) = B$, so that $\mu((a_k))(A) \in (0,1)$, and thus $\mu((a_k))$ is not ergodic for E_0^3, completing the proof.

Let $f_{(a_k)}(x) \in A$ and choose $y \in B$ such that $f_{(a_k)}(x)E_0^3 f_{(a_k)}(y)$. Then, in particular, if $f_{(a_k)}(x) = (x_n), f_{(a_k)}(y) = y_n$, we have $x_{n_0}E_0^3 y_{n_0}$. Now $x_{n_0} = 2\ s_0\ 2\ s_1 \dots$, $y_{n_0} = 2\ t_0\ 2\ t_1 \dots$, where for each i, s_i, t_i are binary sequences of the same length. Let then k be such that for all $i \geq k$, $s_i = t_i$. If $m_j \geq k$, then $t_{m_j} = (y_{n_0}, \dots, y_{n_0+m_j})$ and so $s_{m_j} = t_{m_j} \in T$. Since also $s_{m_j} = (x_{n_0}, \dots, x_{n_0+m_j})$, we have that $(x_{n_0}, x_{n_0+1}, \dots) \in [T]$, i.e., $x \in B$. □

3. Proof of Theorem 2

Ditzen [**D**, page 47] shows that $\mathrm{EQINV}_{\mathbb{Z}}$ is $\mathbf{\Pi}^0_3$ and thus by Theorem 1 it is $\mathbf{\Pi}^0_3$-complete.

4. Proof of Theorem 3

Theorem 3 will follow from the next proposition:

Proposition 4.1. *Let X be a Polish space and let $A \subseteq X$. If $A \leq_W \mathrm{ERG}_{\mathbb{F}_\infty}$, then $A^{\mathbb{N}} (\subseteq X^{\mathbb{N}}) \leq_W \mathrm{ERG}_{\mathbb{F}_\infty}$.*

Proof. Recall that for any countable Borel equivalence relation E, we denote by ERG_E the set of probability Borel measures that are ergodic for E.

Lemma 4.2. *Let E_n be a countable Borel equivalence relation in the Polish space X_n and let μ_n be a probability Borel measure on X_n. Let E_∞ be the following equivalence relation on $X^{\mathbb{N}}$:*

$$(x_n) E_\infty (y_n) \iff \forall n (x_n E_n y_n) \ \& \ \exists m \forall n \geq m (x_n = y_n).$$

Then

$$\prod_n \mu_n \in \mathrm{ERG}_{E_\infty} \iff \forall n (\mu_n \in \mathrm{ERG}_{E_n}).$$

Proof. $\implies$: Put $\mu = \prod_n \mu_n$. Let $A \subseteq X_n$ be Borel and E_n-invariant. Let $B = X_0 \times \cdots X_{n-1} \times A \times X_{n+1} \times \cdots$. Then B is Borel and E_∞-invariant, so $\mu_n(A) = \mu(B) \in \{0, 1\}$.

$\impliedby$: Assume that each μ_n is ergodic for E_n. Let $A \subseteq \prod_n X_n$ be Borel and E_∞-invariant. For each Borel set $B \subseteq \prod_n X_n$, let $\nu(B) = \mu(A \cap B)$. If we can show that for each Borel cylinder $B \subseteq \prod_n X_n$, $\nu(B) = \mu(A)\mu(B)$, then since the class of all Borel sets B with the property that $\nu(B) = \mu(A)\mu(B)$ is closed under complements and countable disjoint unions, by the $\pi - \lambda$ Theorem (see, e.g., [**K**, 10.1, iii)]) it contains all Borel sets, and in particular A, so $\nu(A) = \mu(A) = \mu(A)^2$, thus $\mu(A) \in \{0, 1\}$.

Let then $B = D \times \prod_{i \geq n} X_i$ be a Borel cylinder, where $D \subseteq \prod_{i<n} X_i$. For $y \in \prod_{i \geq n} X_i$, let $A^y = \{(x_i)_{i<n} \in \prod_{i<n} X_i \colon ((x_i)_{i<n}, y)) \in A\}$. Then A^y is $\prod_{i<n} E_i$-invariant.

Claim. $\prod_{i<n} \mu_i \in \mathrm{ERG}_{\prod_{i<n} E_i}$.

Proof. It is enough to consider the case $n = 2$, so let $A \subseteq X_0 \times X_1$ be Borel and $(E_0 \times E_1)$-invariant. Note that for $x_0 \in X_0$ the section $A_{x_0} \subseteq X_1$ is E_1 invariant, so $\mu_1(A_{x_0}) \in \{0, 1\}$. Let $P_i = \{x_0 \colon \mu_1(A_{x_0}) = i\}$, for $i \in \{0, 1\}$. Then each P_i is E_0-invariant. If $\mu_0(P_0) = 0$, then $\mu_0(P_1) = 1$, so $(\mu_0 \times \mu_1)(A) = 1$. If $\mu_0(P_0) = 1$, then $(\mu_0 \times \mu_1)(A) = 1$.

Thus A^y has $\prod_{i<n} \mu_i$-measure 0 or 1. Let

$$C = \{y \in \prod_{i \geq n} X_i \colon (\prod_{i<n} \mu_i)(A^y) = 1\}.$$

Then $\mu(A) = (\prod_{i\geq n}\mu_i)(C)$. Now for $y\in C$, $(\prod_{i<n}\mu_i)(A^y\cap D) = (\prod_{i<n}\mu_i)(D)$ and for $y\notin C$, $(\prod_{i<n}\mu_i)(A_y\cap D) = 0$, so

$$\begin{aligned}\mu(A\cap B) &= \mu(A\cap(D\times\prod_{i\geq n}X_i))\\ &= \int(\prod_{i<n}\mu_i)(A^y\cap D)d(\prod_{i\geq n}\mu_i)(y)\\ &= (\prod_{i<n}\mu_i)(D)\cdot(\prod_{i\geq n}\mu_i)(C)\\ &= \mu(B)\mu(A).\end{aligned}$$

□

Let now E be the equivalence relation on $\mathcal{C}^{\mathbb{F}_\infty}$ induced by the shift action of $\mathbb{F}_\infty$. We have to show that if $A\leq_W \mathrm{ERG}_E$, then $A^{\mathbb{N}}\leq_W \mathrm{ERG}_E$. Let $f\colon X\to P(\mathcal{C}^{\mathbb{F}_\infty})$ be a continuous function witnessing that $A\leq_W \mathrm{ERG}_E$. Define $f_\infty\colon X^{\mathbb{N}}\to P((\mathcal{C}^{\mathbb{F}_\infty})^{\mathbb{N}})$ by $f_\infty((x_n)) = \prod_n f(x_n)$. Then f_∞ is continuous and if E_∞ is as in Lemma 4.2 with $E_n = E$ for each n, then

$$f_\infty((x_n))\in \mathrm{ERG}_{E_\infty} \iff \forall n(f(x_n)\in \mathrm{ERG}_E) \iff (x_n)\in A^{\mathbb{N}}.$$

So $A^{\mathbb{N}}\leq_W \mathrm{ERG}_{E_\infty}$.

Now consider the continuous action of $\bigoplus_n \mathbb{F}_\infty$ on $(\mathcal{C}^{\mathbb{F}_\infty})^{\mathbb{N}}$ given by $(\gamma_n)\cdot(x_n) = (\gamma_n\cdot x_n)$. The equivalence relation it induces is exactly E_∞. Mapping $\mathbb{F}_\infty$ onto $\bigoplus_n\mathbb{F}_\infty$, this gives a continuous action a of $\mathbb{F}_\infty$ on $(\mathcal{C}^{\mathbb{F}_\infty})^{\mathbb{N}}$ for which $\mathrm{ERG}_a = \mathrm{ERG}_{E_\infty}$ and thus $A^{\mathbb{N}}\leq_W \mathrm{ERG}_a$. Noting that $(\mathcal{C}^{\mathbb{F}_\infty})^{\mathbb{N}}$ is homeomorphic to $\mathcal{C}$, we can embed this action to the shift action of $\mathbb{F}_\infty$ on $\mathcal{C}^{\mathbb{F}_\infty}$ and thus $A^{\mathbb{N}}\leq_W \mathrm{ERG}_{\mathbb{F}_\infty}$. □

Using Proposition 4.1, we now complete the proof of Theorem 3 as follows. Let α be least such that $\mathrm{ERG}_{\mathbb{F}_\infty}\in\mathbf{\Pi}^0_\alpha$. By Theorem 1, $\alpha\geq 3$.

Claim. $\mathrm{ERG}_{\mathbb{F}_\infty}$ is $\mathbf{\Pi}^0_\alpha$-complete.

Proof. Let $\mathcal{A} = \{B\subseteq Y\colon Y \text{ Polish, 0-dimensional}, B\leq_W \mathrm{ERG}_{\mathbb{F}_\infty}\}$. Then $\mathcal{A}$ is closed under countable intersections, since if $B_n\in\mathcal{A}, B_n\subseteq Y$, there is a continuous function $f_n\colon Y\to P(\mathcal{C}^{\mathbb{F}_\infty})$ such that $B_n = f_n^{-1}(\mathrm{ERG}_{\mathbb{F}_\infty})$. Put $X = P(\mathcal{C}^{\mathbb{F}_\infty}), A = \mathrm{ERG}_{\mathbb{F}_\infty}$ and let $f\colon Y\to X^{\mathbb{N}}$ be given by $f(y)_n = f_n(y)$. Then f witnesses that $\bigcap_n B_n\leq_W A^{\mathbb{N}}\leq_W A = \mathrm{ERG}_{\mathbb{F}_\infty}$, so $\bigcap_n B_n\in\mathcal{A}$.

Let now $B\in\mathbf{\Pi}^0_\alpha$, $B\subseteq Y$, Y Polish and 0-dimensional. Then $B = \bigcap_n B_n$, where $B_n\in\mathbf{\Sigma}^0_{\alpha_n}$, for some $\alpha_n<\alpha$. By a result of Saint-Raymond (see [**K**, 24.20 and 22.13]) $B_n\leq_W \mathrm{ERG}_{\mathbb{F}_\infty}$, so $B\leq_W \mathrm{ERG}_{\mathbb{F}_\infty}$.

Now, as $\alpha \geq 3$, $\mathrm{EQINV}_{\mathbb{F}_\infty}$ is in $\mathbf{\Pi}^0_\alpha$. Also by Lemma 2.1, if $B \in \mathbf{\Pi}^0_\alpha$, then $B \leq_W \mathrm{ERG}_{\mathbb{F}_\infty} \leq_W \mathrm{EQINV}_{\mathbb{F}_\infty}$, so $\mathrm{EQINV}_{\mathbb{F}_\infty}$ is $\mathbf{\Pi}^0_\alpha$-complete.

Remark 4.3. Note that the only property of $\mathbb{F}_\infty$ that we used in the preceding proof is that it can be mapped onto the direct sum of countably many copies of itself. It follows that Theorem 3 is valid as well for any countable group Γ that has this property.

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