A.S. Kechris and B.D. Miller: Topics in Orbit Equivalence; Corrections and Updates (December 28, 2018)

Page VI, line 16: Add "n odd" in the parenthesis.

Page 3, lines 1,2: The assumption that the F_n are disjoint is not used in this proof.

Page 3, line 4: One can also replace "Since ... that" by "Since X is standard Borel,".

Page 25, Theorem 7.5: The following uniform version of Rokhlin's Lemma for invariant probability measures was proved (see Proposition 7.9) in E. Glasner and B. Weiss, On the interplay between measurable and topological dynamics, in B. Hasselblatt and A. Katok (eds), Handbook of Dynamical Systems, Vol. 1B, Elsevier, (2006), 597–648.

Theorem 1. *Suppose T is an aperiodic Borel automorphism, $n \geq 1$, and $\epsilon > 0$. Then there is a Borel complete section $A \subseteq X$ such that*

(i) $T^i(A) \cap T^j(A) = \emptyset$, *if* $0 \leq i < j < n$,

and for any T-invariant probability measure μ,

(ii) $\mu(X \setminus \bigcup_{i<n} T^i(A)) < \epsilon$, *and*

(iii) $\mu(A) \leq 1/n$.

The following result concerning arbitrary measures implies immediately Theorem 1.

Theorem 2. *Suppose that $m, n \geq 1$, X is a standard Borel space, and $T\colon X \to X$ is an aperiodic Borel automorphism. Then there is a Borel* *complete section $A \subseteq X$ with the property that for every probability measure μ on X, there exist $1 \leq j \leq m$ and $k < n$ such that the set $A_{j,k} = T^{k-2jn}(A)$ has μ-measure at most $1/n$ and $\{T^i(A_{j,k}) \mid i < n\}$ partitions a set of μ-measure at least $1 - 1/m$.*

Proof. Fix a Borel complete section $B \subseteq X$ such that $B \cap T^{-i}(B) = \emptyset$ for all $0 < i < 2mn$, set $B' = \bigcup_{i<n} T^i(B)$, and for all $k \geq 2m$, define $B_k = B \cap (T^{-kn}(B') \setminus \bigcup_{2m \leq j < k} T^{-jn}(B'))$. As E_T is smooth on $B \setminus \bigcup_{k \geq 2m} B_k$ and $X \setminus \bigcup_{k \in \mathbb{N}} T^k(B)$, we can assume that $B = \bigcup_{k \geq 2m} B_k$ and $X = \bigcup_{k \in \mathbb{N}} T^k(B)$. Define $A_k = \bigcup_{j<k} T^{jn}(B_k)$ for all $k \geq 2m$.

To see that the set $A = \bigcup_{k \geq 2m} A_k$ is as desired, suppose that μ is a probability measure on X, and define $A'_k = \bigcup_{i<2n} T^{i+kn}(B_k)$ for all $k \geq 2m$, as well as $A' = \bigcup_{k \geq 2m} A'_k$. Note that if $i < 2n$, $1 \leq j \leq m$, and $k \geq 2m$, then $0 \leq i + kn - 2jn < kn$, so the sets of the form $T^{-2jn}(A')$ for $1 \leq j \leq m$ are pairwise disjoint, thus there exists $1 \leq j \leq m$ such that $\mu(T^{-2jn}(A')) \leq 1/m$. As the sets of the form $T^{k-2jn}(A)$ for $k < n$ are pairwise disjoint, there exists $k < n$ such that $\mu(T^{k-2jn}(A)) \leq 1/n$. As $X = T^{-2jn}(A') \cup \bigcup_{i<n} T^{i+k-2jn}(A)$, it follows that $\mu(\bigcup_{i<n} T^{i+k-2jn}(A)) \geq 1 - \mu(T^{-2jn}(A')) \geq 1 - 1/m$. □

Theorem 1 can be seen also as an immediate consequence of the proof of Theorem 7.5 by applying it to each piece of the ergodic decomposition (see Theorem 3.3) of E_T. In fact, by the same argument, the analog of this uniform Rokhlin Lemma can be proved for the E_T-quasi-invariant probability measures that have a given Radon-Nikodym derivative D (see Section 8) by using the ergodic decomposition theorem of Ditzen, see page 46 of A. Ditzen, Definable equivalence relations on Polish spaces, Ph.D. Thesis, Caltech (1992).

Page 27, Proposition 7.7: Christian Rosendal pointed out the following simpler proof of this proposition, which avoids the need for the Birkhoff ergodic theorem and the assumption that μ is invariant, by replacing the first half of the proof of Proposition 7.7 with the first half of the proof of Theorem 7.5.

Define $A_m, B_m \subseteq X$ exactly as in the proof of Theorem 7.5, and fix $m \in \mathbb{N}$ such that $\mu(B_m) + \mu(X \setminus A) < \epsilon$. Put $A'' = A_m$, and proceed as before: For each $x \in A''$, let $\ell''(x) > 0$ be the least natural number such that $T^{l''(x)}(x) \in A''$, set $k_0(x) = -n$, and recursively define $k_{i+1}(x)$ to be the least natural number such that $T^{k_{i+1}(x)}(x) \in A$ and $k_i(x) + n \leq k_{i+1}(x) \leq \ell''(x) - n$, if such a number exists. Define $B \subseteq X$ by

$$B = \{T^{k_i(x)}(x) : i > 0, x \in A'', \text{ and } k_i(x) \text{ is defined}\},$$

and note that $B \subseteq A$ and $\{T^i(B)\}_{i<n}$ is a pairwise disjoint family which covers $X \setminus (B_m \cup (X \setminus A))$, which is of measure $> 1 - \epsilon$.

Page 28, Remark 7.9: While this follows directly from Proposition 7.7 in the case that μ is invariant, it is false in general. Given $0 < \delta < \epsilon < 0.25$ and a natural number $n \geq 2$, there is an aperiodic Borel automorphism $T : X \to X$, a T-quasi-invariant probability measure μ on X, and a Borel set $A \subseteq X$ of measure $1-\delta$ which does not contain an (ϵ, n)-Rokhlin set of measure $\leq 1/n$. To see this, fix an aperiodic Borel automorphism $T' : X' \to X'$ which admits an invariant probability measure μ', set $X = \{(x, i) : x \in X' \text{ and } i < n\}$, define $T : X \to X$ by

$$T(x, i) = \begin{cases} (x, i+1) & \text{if } i < n-1, \\ (T'(x), 0) & \text{otherwise,} \end{cases}$$

and define μ on X by

$$\mu(B) = (1-\delta)\mu'(\mathrm{proj}_{X'}(B \cap X_0)) + \sum_{1 \leq i < n} \left(\frac{\delta}{n-1}\right) \mu'(\mathrm{proj}_{X'}(B \cap X_i)),$$

where $X_i = X' \times \{i\}$. Now suppose, towards a contradiction, that there is an (ϵ, n)-Rokhlin set $B \subseteq X \times \{0\}$ of measure $\leq 1/n$. Then

$$\mu(B) \leq 1/n \text{ and } \sum_{i<n} \mu(T^i(B)) > 1 - \epsilon.$$

It follows from the definition of μ that for $1 \leq i < n$,

$$\mu(T^i(B)) = \mu(B) \left(\frac{\delta}{n-1}\right) \left(\frac{1}{1-\delta}\right),$$

thus

$$\mu(B) \left(1 + \frac{\delta}{1-\delta}\right) > 1 - \epsilon.$$

It then follows that

$$\frac{1}{n(1-\delta)} > (1-\epsilon),$$

so $2 \leq n < 1/(1-\delta)(1-\epsilon)$, which is impossible, since $\delta < \epsilon < 0.25$.

It should be noted, however, that if we replace the requirement that $\mu(A) > 1 - \epsilon$ with the stronger hypothesis that

$$\mu\left(\bigcap_{i<n} T^{-i}(A)\right) > 1 - \epsilon,$$

then A does contain an (ϵ, n)-Rokhlin set of measure $\leq 1/n$. To see this, set

$$\delta = \epsilon - \mu\left(X \setminus \bigcap_{i<n} T^{-i}(A)\right),$$

appeal to Theorem 7.5 to find a (δ, n)-Rokhlin set $B' \subseteq X$ of measure $\leq 1/n$, and observe that the set $B = A \cap B'$ is as desired.

Page 45, line 6 : $\bigcup_{n\in\mathbb{N}} \bigcap_{m>n} \rightarrow \bigcap_{n\in\mathbb{N}} \bigcup_{m>n}$.

Page 45, line 5 of the proof of 10.5: Open parentheses after "$\forall x \in \mathrm{dom}(F_n)$".

Page 48, line 16-: Add after "identity)", "such that $f_n(x)Ex, \forall x \in S_n$".

Page 49, line 14-: A should also contain 1.

Page 50, line 4-: In the definition of f_n^α, α_n should be $\alpha(n)$.

Page 50, line 2 of Theorem 12.1: Add X after "space".

Page 52, line 6-: Replace C by $X_0 \cup \{x\colon (x,x) \in F_\infty^\alpha\}$; after "$F = E|X_0 \cup F_\infty^\alpha$" add: "to conclude that $\mu(A) = 0$ and thus, as A is a complete section of C, $\mu(C) = 0$."

Page 62, proof or 18.3: Julien Melleray pointed out that one can use the argument in the last paragraph of that proof to show that, for $\mu \in M_f$, we have that $C_\mu(E) < r$ holds iff

$$\exists \epsilon \in \mathbb{Q}^+ \; \forall S \text{ finite} \subseteq \mathbb{N} \; \exists T \text{ finite} \subseteq \mathbb{N} \; [C_\mu(\Theta_T \sqcup \{\theta_i | D(\theta_i, \Theta_T)\}_{i\in S}) \leq r - \epsilon].$$

which directly shows that this condition is Borel on M_f.

Page 84, line 7: Replace "$x \in F$" by "xFy".

Page 89, line 17: After "where" add "$\bar{A}_\theta^0 = A_\theta$ and".

Page 100, line 7: The first $A_{i'}^n$ should be $A_{i'}^{n+1}$.

Page 100, line 11: A^n_{i+1} should be A^{n+1}_i.

Page 102, lines 4 and 5: The exponent of φ_∞ should be n_0 in both cases, not n.

Page 102, line 24: The second $\pi_{n,m}$ should be $\pi_{n,m}(\theta)$.

Page 102, line 14-: $\{\varphi_k)_{k\in K}$ should be $\{\varphi_k\}_{k\in K}$.

Page 102, line 2-: ψ_i should be $\tilde{\psi}_i$.

Page 103, line 3: "extend $\tilde{\varphi}_0$" should be "extend $\tilde{\varphi}$".

Page 106, line 6-: Replace "$E|S_e$" by "$F_e|S_e$".

Page 107, last sentence of Section 28: Replace this sentence by: "Note again that the argument in the proof of 28.8 shows that in 28.14 the following weaker statement is true: either (I)* $C_\mu(E) = 1$ or (II) holds."

Page 108, line 2-: 18.5 should be 18.6.

Page 110, line 2-: $(g \cdot x, h \cdot x)$ should be $(g^{-1} \cdot x, h^{-1} \cdot x)$.

Page 115: Damien Gaboriau has pointed out still another way of seeing that the cost of any infinite amenable group is 1. Suppose, towards a contradiction, that such a group Γ acts freely on a standard Borel space X in a Borel way with invariant ergodic probability measure μ, and $C_\mu(E^X_\Gamma) > 1$. By Lemma 28.12, there is a Borel subtreeing $\mathcal{T} \subseteq E^X_\Gamma$ generating an ergodic equivalence relation $E_\mathcal{T}$ of cost strictly greater than 1. Since subequivalence relations of μ-amenable equivalence relations are μ-amenable, it follows that $E_\mathcal{T}$ is μ-amenable. So from [JKL, 3.23] (which generalizes a result in [A1]), we have that almost every component of $\mathcal{T}$ has at most 2 ends, from which it follows (see, e.g., [JKL, 3.19]) that $E_\mathcal{T}$ is hyperfinite a.e., so has cost 1, a contradiction.

Page 115: After 31.1 add:

Part i) follows from 9.2 and 10.2 and a proof of part ii) is essentially contained in Example 9.4.

Page 121, 35.5: Ioana has extended this result by weakening normality to almost normality and dropping the assumption that N has fixed price.

Pages 123 and 128: Problem 35.7 has been solved by Abért and Nikolov. The answer is negative. See: M. Abért and N. Nikolov, Rank gradient, cost of groups and the rank versus Heegard genus problem, J. Eur. Math. Soc., **14(5)** (2012), 1657–1677.