Chapter 1

Ditzen's effective version of Nadkarni's Theorem

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Dedicated to Katrin Tent

Nadkarni's Theorem asserts that for a countable Borel equivalence relation (CBER) exactly one of the following holds: (1) It has an invariant Borel probability measure or (2) it admits a Borel compression, i.e., a Borel injection that maps each equivalence class to a proper subset of it. An effective version of Nadkarni's Theorem was included in Ditzen's unpublished PhD thesis, where it is shown that if a CBER is effectively Borel, then either alternative (1) above holds or else it admits an effectively Borel compression. In his thesis, Ditzen also proves an effective version of the Ergodic Decomposition Theorem. These notes contain an exposition of these results. We include Ditzen's proof of the Effective Nadkarni's Theorem, and use this construction to provide a different proof of the Effective Ergodic Decomposition Theorem. In addition, we construct a counterexample to show that alternative (1) above does not admit an effective version.

1.1 Introduction

In effective descriptive set theory one often considers the following type of question: Suppose we are given a (lightface) Δ^1_1 structure R on the Baire space $\mathcal{N}$ (like, e.g., an equivalence relation, graph, etc.) and a problem about R that admits a (classical) $\boldsymbol{\Delta}^1_1$ (i.e., Borel) solution. Is there an effective, i.e., Δ^1_1, solution?

For example, consider the case where $R = E$ is a Δ^1_1 equivalence relations which is **smooth**, i.e., admits a Borel function $f\colon \mathcal{N} \to \mathcal{N}$ such that $xEy \iff f(x) = f(y)$. Then it turns out that one can find such a function which is actually Δ^1_1.

One often derives such results via an effective version of a dichotomy theorem, For instance, for the example of smoothness above we have the following classical version of the so-called General Glimm-Effros Dichotomy proved in [6]. Below E_0 is the equivalence relation on the Cantor space $\mathcal{C}$ given by $xE_0y \iff \exists m \forall n \geq m (x(n) = y(n))$.

Theorem 1.1.1 (General Glimm-Effros Dichotomy, see [6]). *Let E be a Borel equivalence relation on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

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(i) E is smooth, i.e., admits a Borel function $f\colon \mathcal{N} \to \mathcal{N}$ such that $xEy \iff f(x) = f(y)$,

(ii) There is a Borel injective function $g\colon \mathcal{C} \to \mathcal{N}$ such that $xE_0y \iff g(x)Eg(y)$.

Now it turns out that the proof of this result in [6] actually gives the following effective version:

Theorem 1.1.2 (Effective General Glimm-Effros Dichotomy, see [6]). *Let E be a Δ^1_1 equivalence relation on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

(i) E admits a Δ^1_1 function $f\colon \mathcal{N} \to \mathcal{N}$ such that $xEy \iff f(x) = f(y)$.

(ii) There is a Borel injective function $g\colon \mathcal{C} \to \mathcal{N}$ such that $xE_0y \iff g(x)Eg(y)$.

From this it is immediate that the smoothness of E is witnessed effectively as mentioned earlier. For more examples of such effectivity results see also the recent paper [11].

In Ditzen's unpublished PhD thesis [2], it is shown that the notion of compressibility of a countable Borel equivalence relation (CBER) is effective, i.e., if a Δ^1_1 CBER on the Baire space $\mathcal{N}$ is compressible, then it admits a Δ^1_1 compression. This follows from an effective version of Nadkarni's Theorem that we state below.

First recall the following standard concepts. A **CBER** E on a standard Borel space X is a Borel equivalence relation all of whose classes are countable. A **compression** of E is an injective map $f : X \to X$ such that for each E-class C we have $f(C) \subsetneqq C$. We say that E is **compressible** if it admits a Borel compression. Finally a Borel probability measure μ on X is **invariant** for E if for any Borel bijection $f\colon X \to X$ with $f(x)Ex, \forall x$, we have that $f_*\mu = \mu$.

We now have:

Theorem 1.1.3 (Nadkarni's Theorem, see [9] and [1]). *Let E be a CBER on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

(i) E is compressible, i.e., admits a Borel compression;

(ii) E admits an invariant probability Borel measure.

We include below Ditzen's proof of the following effective version of Nadkarni's Theorem:

Theorem 1.1.4 (Effective Nadkarni's Theorem [2]). *Let E be a (lightface) Δ^1_1 CBER on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

(i) E admits a Δ^1_1 compression;

(ii) E admits an invariant probability Borel measure.

As a consequence of the proof of the Effective Nadkarni Theorem we also obtain a proof of an effective version of the classical Ergodic Decomposition Theorem (see [3] and [12]). This provides a different proof, for the restricted case of invariant measures, of Ditzen's Effective Ergodic Decomposition Theorem for quasi-invariant measures [2].

First we recall the classical Ergodic Decomposition Theorem. For a CBER E on a standard Borel space X, we let INV_E denote the space of E-invariant probability Borel measures on X. We say $\mu \in \mathrm{INV}_E$ is **ergodic** for E if $\mu(A) \in \{0, 1\}$ for all E-invariant Borel sets $A \subseteq X$, and we let $\mathrm{EINV}_E \subseteq \mathrm{INV}_E$ denote the space of E-ergodic invariant probability Borel measures on X.

Theorem 1.1.5 (Ergodic Decomposition Theorem, see [3] and [12]). *Let E be a CBER on the Baire space $\mathcal{N}$ and suppose that $\mathrm{INV}_E \neq \emptyset$. Then $\mathrm{EINV}_E \neq \emptyset$ and there is a Borel surjection $\pi : \mathcal{N} \to \mathrm{EINV}_E$ such that:*

(i) π is E-invariant;

(ii) if $S_e = \{x : \pi(x) = e\}$, for $e \in \mathrm{EINV}_E$, then $e(S_e) = 1$ and e is the unique E-ergodic invariant probability Borel measure on $E|S_e$;

(iii) for any $\mu \in \mathrm{INV}_E$, $\mu = \int \pi(x) d\mu(x)$.

Nadkarni in [9] noted that his proof of Theorem 1.1.3 can be also used to give a proof of Theorem 1.1.5. We will show below that this argument can also be effectivized.

Let $P(\mathcal{N})$ denote the space of probability Borel measures on $\mathcal{N}$. One can identify a probability Borel measure μ on $\mathcal{N}$ with the map $\varphi_\mu \colon \mathbb{N}^{<\mathbb{N}} \to [0, 1]$, $\varphi_\mu(s) = \mu(N_s)$, where $N_s = \{x \in \mathcal{N} \colon s \subseteq x\}$ (cf. [7, 17.7]). In this way, one may view $P(\mathcal{N})$ as the Π^0_2 subset of $[0, 1]^{\mathbb{N}^{<\mathbb{N}}}$ consisting of all φ satisfying $\varphi(\emptyset) = 1$ and $\varphi(s) = \sum_n \varphi(s^\frown n)$ for all $s \in \mathbb{N}^{<\mathbb{N}}$. Via this identification, we will prove the following effective version of the Ergodic Decomposition Theorem:

Theorem 1.1.6 (Effective Ergodic Decomposition Theorem, see [2]). *Let E be a (lightface) Δ^1_1 CBER on the Baire space $\mathcal{N}$ and suppose that $\mathrm{INV}_E \neq \emptyset$. Then $\mathrm{EINV}_E \neq \emptyset$, and there is a Δ^1_1 E-invariant set $Z \subseteq \mathcal{N}$ and a Δ^1_1 map $\pi : Z \to [0, 1]^{\mathbb{N}^{<\mathbb{N}}}$ such that:*

(i) $E|(\mathcal{N} \setminus Z)$ admits a Δ^1_1 compression, i.e. there is a Δ^1_1 injective map $f : \mathcal{N} \setminus Z \to \mathcal{N} \setminus Z$ such that $f(C) \subsetneq C$ for every E-class $C \subseteq \mathcal{N} \setminus Z$;

(ii) π maps Z onto EINV_E;

(iii) π is E-invariant;

(iv) if $S_e = \{x \in Z \colon \pi(x) = e\}$, for $e \in \mathrm{EINV}_E$, then $e(S_e) = 1$ and e is the unique E-ergodic invariant probability Borel measure on $E|S_e$;

(v) for any $\mu \in \mathrm{INV}_E$, $\mu = \int \pi(x) d\mu(x)$.

In Section 1.4, we will show that there is a Δ^1_1 CBER E on $\mathcal{N}$ that admits an invariant probability Borel measure but does not admit a Δ^1_1 invariant probability measure. It follows that we cannot, in general, make the map π from Theorem 1.1.6 total, because if we could then E would admit a Δ^1_1 invariant probability measure.

1.2 A representation of Δ^1_1 equivalence relations

In this section we will prove a representation of Δ^1_1 CBER that is needed for the proof of Theorem 1.1.4. It can be viewed as a strengthening and effective refinement of the Feldman-Moore Theorem, which asserts that every CBER is obtained from a Borel action of a countable group. Below we use the following terminology:

Definition 1.2.1. A sequence (A_n) of Δ^1_1 subsets of $\mathcal{N}$ is **uniformly** Δ^1_1 if the relation $A \subseteq \mathbb{N} \times \mathcal{N}$ given by

$$A(n.x) \iff x \in A_n,$$

is Δ^1_1. Similarly a sequence (f_n) of partial Δ^1_1 functions $f_n \colon \mathcal{N} \to \mathcal{N}$ (i.e., functions with Δ^1_1 graph) is uniformly Δ^1_1 if the partial function $f \colon \mathbb{N} \times \mathcal{N} \to \mathcal{N}$ given by

$$f(n, x) = f_n(x),$$

is Δ^1_1.

We also say that a countable collection of subsets of $\mathcal{N}$ is uniformly Δ^1_1 if it admits a uniformly Δ^1_1 enumeration. Similarly for a countable set of partial functions.

Theorem 1.2.2 ([2, Section 2.2.1]). *Let E be a Δ^1_1 CBER on the Baire space $\mathcal{N}$. Then*

(1) E is induced by a uniformly Δ^1_1 sequence of (total) involutions, i.e., there is a such a sequence (f_n) with $xEy \iff \exists n(f_n(x) = y)$.

(2) There is a Polish 0-dimensional topology τ on $\mathcal{N}$, extending the standard topology, and a uniformly Δ^1_1 countable Boolean algebra $\mathcal{U}$ of clopen sets in τ, which is a basis for τ and is closed under the group Γ generated by (f_n).

(3) There is a complete compatible metric d for τ such that for every $U \in \mathcal{U}$ and $k > 0$, there is a uniformly Δ^1_1, pairwise disjoint, sequence (U^k_n) with $U^k_n \in \mathcal{U}$, $U = \bigcup_n U^k_n$ and $diam_d(U^k_n) < \frac{1}{k}$, and such that moreover the sequence (U^k_n) is uniformly Δ^1_1 in U, k, n.

Proof. *For (1)*: This follows immediately from the usual proof of the Feldman-Moore Theorem (see [4] or [10, Section 1.2]). So fix below such a sequence (f_n) and consider the corresponding Δ^1_1 action of the group Γ.

For (2), (3): We will first find a topology τ as in (2), which has a uniformly Δ^1_1 countable basis $\mathcal{B}$ of clopen sets closed under the Γ-action, because we can then take $\mathcal{U}$ to be the Boolean algebra generated by $\mathcal{B}$.

For (3) we will find a complete compatible Δ^1_1 metric d for τ (i.e., $d \colon \mathcal{N}^2 \to \mathbb{R}$ is Δ^1_1). Then if $(\mathcal{U}_n)$ is a uniformly Δ^1_1 enumeration of $\mathcal{U}$, we have that

$$A(k, n) \iff diam_d(\mathcal{U}_n) < \frac{1}{k+1}$$

is Π^1_1 and

$$\forall x \in \mathcal{N} \forall k \exists n (n \in A_k \,\&\, x \in \mathcal{U}_n),$$

where $A_k = \{n\colon (k,n) \in A\}$.

So, by the Number Uniformization Theorem for Π^1_1, there is a Δ^1_1 function $f\colon \mathcal{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\forall x \in \mathcal{N} \forall k (f(x,k) \in A_k \ \&\ x \in \mathcal{U}_{f(x,k)}).$$

Since $A'(k,n) \iff \exists x \in \mathcal{N}(n = f(x,k))$ is a Σ^1_1 subset of A, let A'' be Δ^1_1 such that $A' \subseteq A'' \subseteq A$. Since

$$\mathcal{N} \times \mathbb{N} = \bigcup_{(k,n)\in A''} \mathcal{U}_n \times \{k\},$$

we can find a uniformly Δ^1_1 sequence (X^k_n) of sets in $\mathcal{U}$, such that for all $k > 0$ the sequence $(X^k_n)_n$ is a partition of $\mathcal{N}$ of sets with d-diameter less than $\frac{1}{k}$. Finally given any $U \in \mathcal{U}$, let $U^k_n = X^k_n \cap U$.

It thus remains to find τ, d with these properties. We will need first a couple of lemmas.

Lemma 1.2.3. *Let $A \subseteq \mathcal{N}$ be Δ^1_1. Then there is a Polish 0-dimensional topology τ_A on $\mathcal{N}$, which extends the standard topology, has a uniformly Δ^1_1 countable basis consisting of clopen sets containing A, and has a complete compatible Δ^1_1 metric d_A.*

Proof. Let $f\colon \mathcal{N} \to \mathcal{N}$ be computable and let $B \subseteq \mathcal{N}$ be Π^0_1 and such that $f|B$ is injective and $f(B) = A$. Use f to move the (relative) topology of B to A and the standard metric of B to A. Do the same for $\mathcal{N} \setminus A$ and then take the direct sum of these topologies and metrics on $A, \mathcal{N} \setminus A$ to find τ_A, d_A. ■

Lemma 1.2.4. *Let $\mathcal{A} = (A_n)$ be a uniformly Δ^1_1 sequence of subsets of $\mathcal{N}$. Then there is a Polish 0-dimensional topology $\tau_{\mathcal{A}}$ on $\mathcal{N}$, which extends the standard topology, has a uniformly Δ^1_1 countable basis $\mathcal{B}_{\mathcal{A}}$ containing all the sets in $\mathcal{A}$, and has a complete compatible Δ^1_1 metric $d_{\mathcal{A}}$.*

Proof. Consider $\tau_{\mathcal{A}_n}, d_{\mathcal{A}_n}$ as in Lemma 1.2.3. Then put

$$\tau_{\mathcal{A}} = \text{the topology generated by } \bigcup_n \tau_{\mathcal{A}_n}.$$

Then by [7, Lemma 13.3], $\tau_{\mathcal{A}}$ is Polish (and contains the standard topology). A basis for $\tau_{\mathcal{A}}$ consists of all sets of the form

$$U_1 \cap U_2 \cap \cdots \cap U_n,$$

where $U_i \in \mathcal{B}_{A_{j_i}}, 1 \leq I \leq n$, and so it is 0-dimensional with a uniformly Δ^1_1 basis $\mathcal{B}_{\mathcal{A}}$ containing all the sets in $\mathcal{A}$.

Finally, as in the proof of [7, Lemma 13.3] again, a complete compatible metric for $\tau_{\mathcal{A}}$ is

$$d_{\mathcal{A}}(x,y) = \sum_n 2^{-n-1} \cdot \frac{d_{A_n}(x,y)}{1+d_{A_n}(x,y)}.$$

Because of the uniformity in A of the proof of Lemma 1.2.3 this metric is also Δ^1_1. ■

We finally find τ, d. To do this we recursively define a sequence of Polish 0-dimensional topologies $\tau_0, \tau_1, \ldots$ on $\mathcal{N}$, extending the standard topology, and uniformly Δ^1_1 countable bases $\mathcal{B}_n$ for τ_n and complete compatible Δ^1_1 metrics d_n for τ_n, *all uniformly in n as well*, and such that $\Gamma \cdot \mathcal{B}_n \subseteq \mathcal{B}_{n+1}$.

For $n = 0$, let $\tau_0, d_0, \mathcal{B}_0$ be the standard topology, metric and basis for $\mathcal{N}$.

Given $\tau_n, d_n, \mathcal{B}_n$, consider $\Gamma \cdot \mathcal{B}_n$ and use Lemma 1.2.4 to define $\tau_{n+1}, \mathcal{B}_{n+1} \supseteq \Gamma \cdot \mathcal{B}_n, d_{n+1}$. The uniformity in n is clear from the construction.

Finally let τ be the topology generated by $\bigcup_n \tau_n$. It is 0-dimensional, Polish, with basis the sets of the form

$$U_1 \cap U_2 \cap \cdots \cap U_n,$$

with $U_i \in \mathcal{B}_{j_i}, 1 \leq i \leq n$, so this is a uniformly Δ^1_1 countable basis $\mathcal{B}$ consisting of clopen sets. Also clearly for any $\gamma \in \Gamma$,

$$\gamma \cdot (U_1 \cap U_2 \cap \cdots \cap U_n) = \gamma \cdot U_1 \cap \gamma \cdot U_2 \cap \cdots \cap \gamma \cdot U_n,$$

where $\gamma \cdot U_i \in \mathcal{B}_{j_i+1}$, thus $\gamma \cdot (U_1 \cap U_2 \cap \cdots \cap U_n) \in \mathcal{B}$ as well. Finally, as before, a complete compatible Δ^1_1 metric for τ is

$$d(x,y) = \sum_n 2^{-n-1} \cdot \frac{d_n(x,y)}{1+d_n(x,y)}$$

and the proof is complete. ■

1.3 Proof of Effective Nadkarni

In this section we show, using the representation of Δ^1_1 CBER constructed in Section 1.2, that we can effectivize the proof of Nadkarni's Theorem. Our proof follows the exposition in [2, Section 2.2.3]; see also the presentations of the classical proof in [1] or [10].

The classical proof of Nadkarni's Theorem proceeds as follows. Fix a CBER E on $\mathcal{N}$. We first define a way to compare the "size" of sets. For Borel sets $A, B \subseteq \mathcal{N}$ we write $A \sim_B B$ if there is a Borel bijection $g\colon A \to B$ with $xEg(x), \forall x \in A$. We write $A \prec_B B$ if there is some $B' \subseteq B$ with $A \sim_B B'$ and $[B]_E = [B \setminus B']_E$, and $A \approx_B nB$ if we can partition A into pieces $A_0, \ldots, A_n$ so that $A_i \sim_B B$ for $i < n$ and $A_n \prec_B B$. One thinks of $A \approx_B nB$ to mean that A is about n times the size of B. In particular, if $A \approx_B nB$ and μ is an E-invariant probability Borel measure, then $n\mu(B) \leq \mu(A) \leq (n+1)\mu(B)$.

Note that E is compressible iff $\mathcal{N} \prec_B \mathcal{N}$. More generally, we say that $A \subseteq \mathcal{N}$ is *compressible* if $A \prec_B A$, i.e., if the equivalence relation $E|A$ is compressible.

Next we construct a *fundamental sequence* for E, i.e., a decreasing sequence (F_n) of Borel sets such that $F_0 = \mathcal{N}$ and $F_{n+1} \sim_B F_n \setminus F_{n+1}$. Each F_n is a complete section for E, and is a piece of $\mathcal{N}$ of "size" 2^{-n}, in the sense that $\mathcal{N} \approx_B 2^n F_n$ and $\mu(F_n) = 2^{-n}$ for all E-invariant probability Borel measures μ. It follows that if $A \approx_B kF_n$ then $k2^{-n} \leq \mu(A) \leq (k+1)2^{-n}$ for any E-invariant probability Borel measure μ.

We then use the relative size of A with respect to the F_n to approximate what the measure of A would be with respect to some E-invariant probability Borel measure. To do this, we construct, for all m, a partition $[A]_E = \bigsqcup_{n\leq\infty} Q_n^{A,m}$ of $[A]_E$ into E-invariant Borel pieces such that $Q_\infty^{A,m}$ admits a Borel compression and $A \cap Q_n^{A,m} \approx_B n(F_m \cap Q_n^{A,m})$ for $n < \infty$. We define the *fraction function* $[A/F_m]$ by setting $[A/F_m](x) = n$ if $x \in Q_n^{A,m}$ or if $n = 0 \ \ \& \ \ x \notin [A]_E$, and let the *local measure function* $m(A,x) = \lim_{m\to\infty} \frac{[A/F_m](x)}{[\mathcal{N}/F_m](x)}$. We show that $m(A,x)$ is well-defined modulo an E-invariant compressible set, meaning there is an E-invariant set $C \subseteq \mathcal{N}$ admitting a Borel compression and such that $m(A,x)$ is well-defined when $x \notin C$. We also show that for any partition $A = \bigsqcup_n A_n$ into Borel pieces we have $m(A,x) = \sum_n m(A_n,x)$ modulo an E-invariant compressible set, and if $A \sim B$ then $m(A,x) = m(B,x)$ modulo an E-invariant compressible set.

Finally, we show that the local measure function $m(\cdot,x)$ defines an E-invariant probability Borel measure, for all $x \in \mathcal{N} \setminus C$, where $C \subseteq \mathcal{N}$ is some E-invariant compressible set. To see this, we fix a Borel action $\Gamma \curvearrowright \mathcal{N}$ of a countable group Γ inducing E, a zero-dimensional Polish topology τ on $\mathcal{N}$ extending the usual one in which the action $\Gamma \curvearrowright \mathcal{N}$ is continuous, a complete compatible metric d for τ and a countable Boolean algebra of clopen-in-τ sets closed under the Γ action forming a basis for τ, and satisfying additionally that for every $U \in \mathcal{U}$ and $k > 0$ there is a pairwise disjoint sequence (U_n^k) of sets in $\mathcal{U}$ with $U = \bigcup_n U_n^k$ and $diam_d(U_n^k) < \frac{1}{k}$. For each $U \in \mathcal{U}$, $k > 0$ we fix such a sequence. Since the countable union of Borel E-invariant compressible sets is itself a Borel E-invariant compressible set, it follows that there is an E-invariant compressible set $C \subseteq \mathcal{N}$ such that for $x \notin C$ we have $m(U,x) = \sum_n m(U_n^k, x)$ for $U \in \mathcal{U}$, $k > 0$, $m(U \cup V, x) = m(U,x) + m(V,x)$ for $U, V \in \mathcal{U}$ disjoint, and $m(U,x) = m(\gamma U, x)$ for $U \in \mathcal{U}, \gamma \in \Gamma$. Using this, we show that for $x \notin C$ there is an E-invariant probability Borel measure μ with $\mu(U) = m(U,x)$ for $U \in \mathcal{U}$. It follows that either $C = \mathcal{N}$, in which case E is compressible, or E admits an invariant probability Borel measure.

In order to prove the effective version of Nadkarni's Theorem, we will show that the classical proof outlined above can be effectivized using the representation in Section 1.2.

For the remainder of this section, we fix a Δ^1_1 CBER E on $\mathcal{N}$ and a uniformly Δ^1_1 sequence of (total) involutions (γ_n) inducing E, as in Theorem 1.2.2(1). Moreover, we assume, without loss of generality, that E is **aperiodic**, meaning that every E-class is infinite, because if $C \subseteq \mathcal{N}$ were a finite E-class then the uniform measure on C would be an E-invariant probability Borel measure.

(A) Comparing the "size" of sets.

We begin by defining a way to compare the "size" of Δ^1_1 sets. The notation we use is the same as the notation typically used for the equivalent classical notions (cf. [10, Definition 2.2.4, Section 2.3]), which we denoted with the subscript B above. In this paper, these notions will *always* refer to the effective definitions below.

Definition 1.3.1. Let $A, B \subseteq \mathcal{N}$ be Δ^1_1.

(1) We write $A \sim B$ if there is a Δ^1_1 bijection $f\colon A \to B$ and such that $xEf(x), \forall x \in A$. If f is such a function we write $f\colon A \sim B$.

(2) We write $A \preceq B$ if $A \sim B'$ for some Δ^1_1 subset $B' \subseteq B$. If f is such a function we write $f\colon A \preceq B$.

(3) We write $A \prec B$ if there is some $f\colon A \preceq B$ such that $[B \setminus f(A)]_E = [B]_E$. If f is such a function we write $f\colon A \prec B$.

(4) We say A **admits a** Δ^1_1 **compression** if $A \prec A$, and if $f\colon A \prec A$ then we call f a Δ^1_1 compression of A.

(5) We write $A \preceq nB$ if there are Δ^1_1 sets $A_i, i < n$ such that $A = \bigcup_{i<n} A_i$ and $A_i \preceq B$ for $i < n$. Note that $A \preceq 1B \iff A \preceq B$.

(6) We write $A \prec nB$ if in the previous definition there is some $i < n$ for which $A_i \prec B$. Note that $A \prec 1B \iff A \prec B$.

(7) We write $A \succeq nB$ if there are pairwise disjoint Δ^1_1 sets $B_i \subseteq A, i < n$ such that $B_i \sim B$.

(8) We write $A \approx nB$ if there is a partition $A = \bigsqcup_{i<n} B_i \sqcup R$ into Δ^1_1 pieces such that $B_i \sim B$ and $R \prec B$. In particular, $A \approx 0B \iff A \prec B$. Note that $A \approx nB$ implies that $A \succeq nB$ and $A \prec (n+1)B$.

We also let $\mathscr{H}$ denote the set of all E-invariant Δ^1_1 subsets $C \subseteq \mathcal{N}$ that admit a Δ^1_1 compression.

Lemma 1.3.2. *(1) Let $A \subseteq \mathcal{N}$ be Δ^1_1. If $A \prec A$ then $[A]_E \prec [A]_E$.*

(2) Let $(A_n), (B_n)$ be uniformly Δ^1_1 families of E-invariant sets and let (f_n) be a uniformly Δ^1_1 sequence of maps satisfying $f_n\colon A_n \prec B_n$. Then $\bigcup_n A_n \prec \bigcup_n B_n$. The same holds when $\prec$ is replaced by $\preceq$ or $\sim$, or if these are sequences of pairwise disjoint but not necessarily E-invariant sets.

(3) Let $A, B, C \subseteq \mathcal{N}$ be Δ^1_1. If $A \succeq nB$ and $C \preceq mB$ for some $m \leq n$, then $C \preceq A$. If additionally $C \prec mB$ then $C \prec A$.

Proof. (1) Let $f\colon A \prec A$ and let $g(x) = f(x)$ for $x \in A$, $g(x) = x$ for $x \in [A]_E \setminus A$. Then $g\colon [A]_E \prec [A]_E$.

(2) For $x \in \bigcup_n A_n$ set $f(x) = f_n(x)$ where n is least with $x \in A_n$. Then $f\colon \bigcup_n A_n \prec \bigcup_n B_n$.

(3) Let $A_i, i < n$ be pairwise disjoint Δ^1_1 subsets of A, $f_i\colon A_i \sim B$ for $i < n$, $C_j, j < m$ be Δ^1_1 sets covering C and $g_j\colon C_j \preceq B$ for $j < m$. Define

$$h(x) = f_j^{-1} \circ g_j(x) \text{ for } j \text{ least with } x \in C_j.$$

Then $h\colon C \preceq A$, and if $g_j\colon C_j \prec B$ then, letting $C' = C_j \setminus \bigcup_{k<j} C_k$, we have

$$[A \setminus h(C)]_E \supseteq ([A]_E \setminus [B]_E) \cup [B \setminus g_j(C')]_E = ([A]_E \setminus [B]_E) \cup [B]_E = [A]_E,$$

so $f\colon C \prec A$. ■

(B) Fundamental sequences.

Definition 1.3.3. A **uniformly Δ^1_1 fundamental sequence** for E is a uniformly Δ^1_1 decreasing sequence (F_n) of sets and a uniformly Δ^1_1 sequence (f_n) of maps such that $F_0 = \mathcal{N}$ and $f_n\colon F_{n+1} \sim F_n \setminus F_{n+1}$ for all n.

Lemma 1.3.4. *Let $X \subseteq \mathcal{N}$ be a Δ^1_1 set on which $E|X$ is aperiodic. Then there is a partition $X = A \sqcup B$ of X into Δ^1_1 pieces such that $A \sim B$. In particular, $E|A, E|B$ are also aperiodic.*

Proof. Let $<$ be a Δ^1_1 linear order on $\mathcal{N}$ (for example the lexicographic order) and let $x \in A_n \iff x < \gamma_n x$. Define recursively the sets

$$\tilde{A}_n = \{x \in X \cap A_n \colon x, \gamma_n x \in X \setminus \bigcup_{i<n} (\tilde{A}_i \cup \gamma_i \tilde{A}_i)\}.$$

Let $A = \bigsqcup_n \tilde{A}_n$ and define $f = \bigcup_n \gamma_n | \tilde{A}_n \colon A \to X$. Because of the uniformity of this construction, A, f are Δ^1_1. It is easy to see that f is injective and that $f(A) \cap A = \emptyset$, so in particular that $f\colon A \sim f(A)$.

We claim that $A \cup f(A)$ omits at most one point from each $E|X$-class. To see this, let $x < y \in X$ and suppose that xEy. Let $\gamma_n x = y$. If $x, y \notin \bigcup_{i<n} (\tilde{A}_i \cup \gamma_i \tilde{A}_i)$, then by definition we have $x, y \in \tilde{A}_n \cup \gamma_n \tilde{A}_n \subseteq A \cup f(A)$.

Now let $T = X \setminus (A \cup f(A)), Y = X \cap [T]_E, Z = X \setminus [T]_E$. Then T is a traversal of $E|Y$ and $f|(A \cap Z)\colon A \cap Z \sim f(A) \cap Z$. Thus it remains to prove the lemma for $E|Y$. In this case, using T and the sequence (γ_n), one can enumerate each $E|Y$-class, and since these are infinite we can take A (resp. B) to be the even (resp. odd) elements of this enumeration. ■

Proposition 1.3.5. *There exists a uniformly Δ^1_1 fundamental sequence for E.*

Proof. We construct the sequences recursively. Let $F_0 = \mathcal{N}$ and recursively apply Lemma 1.3.4 to get F_{n+1} and $f_n\colon F_{n+1} \sim F_n \setminus F_{n+1}$. Uniformity of these sequences follows from the uniformity in the proof of Lemma 1.3.4. ■

For the remainder of this section, we fix a uniformly Δ^1_1 fundamental sequence (F_n) for E.

(C) Decompositions of Δ^1_1 sets.

Lemma 1.3.6. *Let $A, B \subseteq \mathcal{N}$ be Δ^1_1 and let $Z = [A]_E \cap [B]_E$. There is a partition $Z = P \sqcup Q$ of Z into E-invariant uniformly Δ^1_1 sets such that $A \cap P \prec B \cap P$ and $B \cap Q \preceq A \cap Q$.*

Proof. Define recursively the sets

$$A_n = \{x \in A \setminus \bigcup_{i<n} A_i \colon \gamma_n x \in B \setminus \bigcup_{i<n} B_i\}, B_n = \gamma_n A_n.$$

Let $\tilde{A} = \bigcup_n A_n, \tilde{B} = \bigcup_n B_n$ and $f = \bigcup_n \gamma_n | A_n$. By the uniformity of this construction, $\tilde{A}, \tilde{B}, f$ are all Δ^1_1, so that $f\colon \tilde{A} \sim \tilde{B}$. If we set $P = Z \cap [B \setminus \tilde{B}]_E, Q = Z \setminus P$ then it is easy to see that $A \cap P \subseteq \tilde{A}, B \cap Q \subseteq \tilde{B}$ and hence that $f|(A \cap P)\colon A \cap P \prec B \cap P$ and $f^{-1}|(B \cap Q)\colon B \cap Q \preceq A \cap Q$. ■

Proposition 1.3.7. *Let $A, B \subseteq \mathcal{N}$ be Δ^1_1 and let $Z = [A]_E \cap [B]_E$. There exists a partition $Z = \bigsqcup_{n \leq \infty} Q_n$ of Z into E-invariant Δ^1_1 pieces such that $A \cap Q_n \approx n(B \cap Q_n)$ for $n < \infty$ and $Q_\infty \in \mathscr{H}$.*

Proof. We recursively construct sequences of sets

$$A_n, B_n, \tilde{P}_n, \tilde{Q}_n, f_n, g_n, \tilde{B}_n, Q_n, R_n, B^i_n, f^i_n$$

for $i < n$ such that $A \cap Q_n = \bigsqcup_{i<n} B^i_n \sqcup R$ for $n < \infty$, $f^i_n\colon B^i_n \sim B \cap Q_n$ for $i < n < \infty$, and $f_n\colon R_n \prec B \cap Q_n$ for $n < \infty$.

First we let $A_0 = A, B_0 = B$. We apply Lemma 1.3.6 to these sets to get $\tilde{P}_0, \tilde{Q}_0, f_0, g_0$ and $\tilde{B}_0$ satisfying

$$f_0\colon A_0 \cap \tilde{P}_0 \prec B_0 \cap \tilde{P}_0, \quad g_0\colon B_0 \cap \tilde{Q}_0 \preceq A_0 \cap \tilde{Q}_0, \quad \tilde{B}_0 = \mathrm{Im}(g_0).$$

Define $Q_0 = \tilde{P}_0, R_0 = A_0 \cap Q_0$.

Now let $n > 0$ and suppose we have already constructed

$$A_k, B_k, \tilde{P}_k, \tilde{Q}_k, f_k, g_k, \tilde{B}_k, Q_k, R_k, B^i_k, f^i_k$$

for all $i < k < n$. Let $A_n = (A_{n-1} \cap \tilde{Q}_{n-1}) \setminus \tilde{B}_{n-1}, B_n = B \cap \tilde{Q}_{n-1}$. Apply Lemma 1.3.6 to A_n, B_n to get $\tilde{P}_n, \tilde{Q}_n, f_n, g_n, \tilde{B}_n$ such that

$$f_n\colon A_n \cap \tilde{P}_n \prec B_n \cap \tilde{P}_n, \quad g_n\colon B_n \cap \tilde{Q}_n \preceq A_n \cap \tilde{Q}_n, \quad \tilde{B}_n = \mathrm{Im}(g_n).$$

Define $Q_n = \tilde{Q}_{n-1} \setminus \tilde{Q}_n$, $R_n = A_n \cap Q_n$, $B^i_n = \tilde{B}_i \cap Q_n$, $f^i_n = (g_i)^{-1}|B^i_n$.

By uniformity of this construction it is clear that these sequences are uniformly Δ^1_1. Additionally, $A \cap Q_n \approx n(B \cap Q_n)$ for $n < \infty$.

Now let $Q_\infty = Z \setminus \bigcup_n Q_n = \bigcap_n \tilde{Q}_n$. The sets $\tilde{B}_n$ are pairwise disjoint and $g_n \colon B \cap \tilde{Q}_n \sim \tilde{B}_n$ for all n. Therefore, if we define $B^n_\infty = \tilde{B}_n \cap Q_\infty$, $g^n_\infty = g_n|(B \cap Q_\infty)$ and $g^{n,m}_\infty = g^m_\infty \circ (g^n_\infty)^{-1}$, we have that the B^n_∞ are pairwise disjoint and $g^{n,m}_\infty \colon B^n_\infty \sim B^m_\infty$. Let $B_\infty = \bigcup_n B^n_\infty$ and $g_\infty = \bigcup_n g^{n,n+1}_\infty$. Then B_∞, g_∞ are Δ^1_1 and $g_\infty \colon B_\infty \prec B_\infty$. Since $[B_\infty]_E = [B^0_\infty]_E = [B \cap Q_\infty]_E = Q_\infty$, Q_∞ admits a Δ^1_1 compression by Lemma 1.3.2(1). ■

Notation 1.3.8. For Δ^1_1 sets $A, B \subseteq \mathcal{N}$, we let $Q^{A,B}_n$, $n \leq \infty$ be the decomposition of $[A]_E \cap [B]_E$ constructed in Proposition 1.3.7.

(D) The fraction functions.

Definition 1.3.9. We associate to all Δ^1_1 sets $A, B \subseteq \mathcal{N}$ a **fraction function** $[A/B] : \mathcal{N} \to \mathbb{N}$ defined by

$$\left[\frac{A}{B}\right](x) = \begin{cases} n & \text{if } x \in Q^{A,B}_n \text{ for some } n \leq \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.3.10. *Let $A, A_0, A_1, A_2, B, S \subseteq \mathcal{N}$ be Δ^1_1.*

(1) If xEy then $[A/B](x) = [A/B](y)$.

(2) If $A_0 \preceq A_1$ then there is some $C \in \mathscr{H}$ such that $[A_0/B](x) \leq [A_1/B](x)$ for $x \notin C$.

(3) If $A_0 \sim A_1$ then there is some $C \in \mathscr{H}$ such that $[A_0/B](x) = [A_1/B](x)$ for $x \notin C$.

(4) If S is E-invariant then there is some $C \in \mathscr{H}$ such that for $x \in S \setminus C$ we have $[A/B](x) = [(A \cap S)/B](x)$.

(5) If A_0, A_1 are disjoint then there is some $C \in \mathscr{H}$ such that for $x \notin C$,

$$[A_0/B] + [A_1/B] \leq [(A_0 \cup A_1)/B] \leq [A_0/B] + 1 + [A_1/B] + 1.$$

(6) If A_1 is an E-complete section then there is some $C \in \mathscr{H}$ such that for $x \notin C$,

$$[A_0/A_1][A_1/A_2] \leq [A_0/A_2] < ([A_0/A_1] + 1)([A_1/A_2] + 1).$$

(7) There is some $C \in \mathscr{H}$ such that $[F_n/F_{n+m}] = 2^m$ holds for all $m, n \in \mathbb{N}, x \notin C$.

(8) There is some $C \in \mathscr{H}$ such that for all $x \in [A]_E \setminus C$ we have $[A/F_n](x) \to \infty$.

(9) The set $Y = \{x : [A_0/B](x) < [A_1/B](x)\}$ is Δ^1_1 and E-invariant and $A_0 \cap Y \preceq A_1 \cap Y$.

Proof. (1) This is clear, as the sets $Q^{A,B}_n$ are E-invariant.

(2) Let $C_{n,m} = Q^{A_0,B}_n \cap Q^{A_1,B}_m$ for $m < n$. Then $A_0 \cap C_{n,m} \approx n(B \cap C_{n,m})$ and $A_1 \cap C_{n,m} \approx m(B \cap C_{n,m})$ so by Lemma 1.3.2(3) and our assumption we have $A_0 \cap$

$C_{n,m} \preceq A_1 \cap C_{n,m} \prec A_0 \cap C_{n,m}$. By Lemma 1.3.2(2) and the uniformity of the proofs of Proposition 1.3.7 and Lemma 1.3.2(3), $C = \bigcup_{m<n} C_{n,m} \in \mathscr{H}$, and $[A_0/B](x) \leq [A_1/B](x)$ for $x \notin C$.

(3) This follows from (2).

(4) As in the proof of (2), it suffices to show that $C = S \cap Q_k^{A,B} \cap Q_l^{A \cap S,B}$ admits a Δ^1_1 compression (in a uniform way) for $k \neq l$. But

$$A \cap C \approx k(B \cap C) \text{ and } A \cap C \approx l(B \cap C)$$

by E-invariance of C, so by Lemma 1.3.2(1),(3) C admits a Δ^1_1 compression.

(5) Let $C = C_{i,j,k} = Q_i^{A_0,B} \cap Q_j^{A_1,B} \cap Q_k^{A_2,B}$. Then (5) fails to hold exactly when $x \in C_{i,j,k}$ for $k < i + j$ or $k > i + 1 + j + 1$. Therefore, as in the proof of (2), it suffices to show that $C_{i,j,k}$ admits a Δ^1_1 compression (in a uniform way) for such i, j, k.

Now we know that $A_0 \cap C \approx i(B \cap C)$, $A_1 \cap C \approx j(B \cap C)$, $A_2 \cap C \approx k(B \cap C)$ by E-invariance of C. If $k < i + j$ then $(A_0 \cup A_1) \cap C \prec (i + j)(B \cap C)$ and (since A_0, A_1 are disjoint) we have $(A_0 \cap C) \cup (A_1 \cap C) \succeq (i + j)(B \cap C)$. Thus by Lemma 1.3.2(1),(3) $C = [(A_0 \cup A_1) \cap C]_E$ admits a Δ^1_1 compression. On the other hand, if $k > i + 1 + j + 1$ then $(A_0 \cap C) \cup (A_1 \cap C) \prec (i + 1 + j + 1)(B \cap C)$ and $(A_0 \cup A_1) \cap C \succeq k(B \cap C)$, so again C admits a Δ^1_1 compression.

(6) If $x \notin [A_0]_E \cup [A_2]_E$ then this clearly holds. Thus if $C = C_{k,l,m} = Q_k^{A_0,A_1} \cap Q_l^{A_1,A_2} \cap Q_m^{A_0,A_2}$ then (6) fails to hold exactly when $x \in C_{k,l,m}$ for $m < kl$ or $m \geq (k + 1)(l + 1)$. Therefore, as in the proof of (2), it suffices to show that these sets admit a Δ^1_1 compression (in a uniform way).

Since $A_0 \cap C \approx k(A_1 \cap C)$ and $A_1 \cap C \approx l(A_2 \cap C)$ we have that $A_0 \cap C \succeq kl(A_2 \cap C)$. Also, $A_0 \cap C \approx m(A_2 \cap C)$, so if $kl > m$ then by Lemma 1.3.2(1),(3) we are done. On the other hand, if $m \geq (k + 1)(l + 1)$ then $A_0 \cap C \succeq (k + 1)(l + 1)(A_2 \cap C)$, and since $A_1 \cap C \approx l(A_2 \cap C)$ one easily sees that $A_0 \cap C \succeq (k + 1)(A_1 \cap C)$. Thus by Lemma 1.3.2(1),(3) we are done.

(7) Again it suffices to show that $Q_k^{F_n,F_{n+m}}$ admits a Δ^1_1 compression in a uniform way for $k \neq 2^m$. When $k = \infty$ this is clear. Otherwise, one easily sees by definition of the fundamental sequence that $F_n \approx 2^m F_{n+m}$, and moreover there is a uniformly Δ^1_1 sequence of witnesses to this. It follows that $F_n \cap Q_k^{F_n,F_{n+m}} \approx 2^m(F_{n+m} \cap Q_k^{F_n,F_{n+m}})$ and $F_n \cap Q_k^{F_n,F_{n+m}} \approx k(F_{n+m} \cap Q_k^{F_n,F_{n+m}})$, so when $k \neq 2^m$ this follows from Lemma 1.3.2(1),(3).

(8) Let C_0 be the set constructed in (7), $C(A_0, A_1, A_2)$ be the set constructed in (6), $C_1 = \bigcap_n Q_0^{A,F_n}$ and

$$C_2 = \bigcup_{n,m} C(A, F_n, F_{n+m}) \cup \bigcup_n C(F_0, F_n, A).$$

Let $C = C_0 \cup C_1 \cup C_2$. If $x \in [A]_E \setminus C$ then there is some n for which $[A/F_n](x) \neq 0$, so for all m we have

$$[A/F_{n+m}](x) \geq [A/F_n](x)[F_n/F_{n+m}](x) \geq 2^m,$$

and therefore $[A/F_n](x) \to \infty$. By the uniformity of the proofs of (6), (7) and Proposition 1.3.7, C is Δ^1_1, so it remains to show that it admits a Δ^1_1 compression. By the uniformity of the proofs of (6), (7) and Lemma 1.3.2(2), it suffices to show that $C_1 \setminus (C_0 \cup C_2)$ admits a Δ^1_1 compression.

First we show that $C_1 \cap \bigcup_n Q_0^{F_n,A}$ admits a Δ^1_1 compression. For this it suffices to show that $C_1 \cap Q_0^{F_n,A}$ admits a Δ^1_1 compression for all n (in a uniform way), by Lemma 1.3.2(2). But by definition and E-invariance we have

$$F_n \cap C_1 \cap Q_0^{F_n,A} \prec A \cap C_1 \cap Q_0^{F_n,A} \prec F_n \cap C_1 \cap Q_0^{F_n,A},$$

so $F_n \cap C_1 \cap Q_0^{F_n,A}$ admits a Δ^1_1 compression, and since F_n is a complete section we are done by Lemma 1.3.2(1).

Next we consider $C' = C_1 \setminus (C_0 \cup C_2 \cup \bigcup_n Q_0^{F_n,A})$. For any $x \in C', n \in \mathbb{N}$, we have

$$[F_0/A](x) \geq [F_0/F_n](x)[F_n/A](x) \geq 2^n,$$

so $[F_0/A](x) = \infty$ and $x \in Q_\infty^{F_0,A}$. Thus $C' \subseteq Q_\infty^{F_0,A}$ admits a Δ^1_1 compression.

(9) This set is clearly Δ^1_1 and it is E-invariant by (1). Next note that $Y \subseteq [B]_E \setminus Q_\infty^{A_0,B}$ so we can decompose Y into $Y_0 = Y \setminus [A_0]_E$ and $Y_1 = Y \cap [A_0]_E = \bigcup_n (Y \cap Q_n^{A_0,B})$. Since $Y_0 \cap A_0 = \emptyset$ we clearly have $Y_0 \cap A_0 \preceq Y_0 \cap A_1$, so it remains to show that $Y_1 \cap A_0 \preceq Y_1 \cap A_1$. But by Lemma 1.3.2(3) we have that $A_0 \cap Q_m^{A_0,B} \cap Q_n^{A_1,B} \preceq A_1 \cap Q_m^{A_0,B} \cap Q_n^{A_1,B}$ for $m < n$, so by Lemma 1.3.2(2) we are done. ■

(E) Local measures.

Proposition 1.3.11. *Let $A \subseteq \mathcal{N}$ be Δ^1_1. Then there is some $C \in \mathscr{H}$ such that*

$$\lim_n \frac{[A/F_n](x)}{[\mathcal{N}/F_n](x)}$$

exists for $x \notin C$, and the limit is zero for $x \notin [A]_E \cup C$ and is non-zero and finite for $x \in [A]_E \setminus C$.

Proof. Let $C_0(A_0, A_1, A_2), C_1, C_2(A)$ be the sets we have constructed in the proofs of Lemma 1.3.10(6)(7)(8), respectively, and take $C = \bigcup_{n,m} C_0(A, F_n, F_{n+m}) \cup C_1 \cup C_2(A) \cup \bigcup_n Q_\infty^{A,F_n}$. By Lemma 1.3.2(2) and the uniformity of Lemma 1.3.10, $C \in \mathscr{H}$. If $x \notin [A]_E \cup C$ then $[A/F_n](x) = 0$ and $[\mathcal{N}/F_n](x) = 2^n$ for all n, so the limit exists and is zero.

Now suppose that $x \in [A]_E \setminus C$. Then $[F_n/F_{n+m}](x) = 2^m$ for all $m, n \in \mathbb{N}$, and

$$[A/F_{n+m}](x) \leq ([A/F_n](x) + 1)([F_n/F_{n+m}](x) + 1),$$

so

$$\limsup_{m\to\infty} \frac{[A/F_{n+m}](x)}{[\mathcal{N}/F_{n+m}](x)} \leq \frac{[A/F_n](x) + 1}{[\mathcal{N}/F_n](x)}.$$

Thus the limit exists and is finite at x. To see that the limit is non-zero at x, note that $[A/F_{n+m}](x) \geq [A/F_n](x)[F_n/F_{n+m}](x)$ for all $m, n \in \mathbb{N}$, so

$$\liminf_{m\to\infty} \frac{[A/F_{n+m}](x)}{[\mathcal{N}/F_{n+m}](x)} \geq \frac{[A/F_n](x)}{[\mathcal{N}/F_n](x)}$$

for all n, and since $[A/F_n](x) \to \infty$ this lower bound must be non-zero for some n. ■

Definition 1.3.12. Let $A \subseteq \mathcal{N}$ be Δ^1_1 and let $C_A \in \mathscr{H}$ be the set constructed in the proof of Proposition 1.3.11. We associate to A the **local measure function** $m(A, \cdot)\colon \mathcal{N} \setminus C_A \to \mathbb{R}$ defined by

$$m(A, x) = \lim_n \frac{[A/F_n](x)}{[\mathcal{N}/F_n](x)}.$$

Note that the local measure function is Δ^1_1, uniformly in A.

Lemma 1.3.13. *Let $A, B, S \subseteq \mathcal{N}$ be Δ^1_1.*

(1) If xEy then $m(A, x) = m(A, y)$ for $x, y \notin C_A$.

(2) Let $Y = \{x \in \mathcal{N} \setminus (C_A \cup C_B)\colon m(A, x) < m(B, x)\}$. Then Y is Δ^1_1, E-invariant and $A \cap Y \preceq B \cap Y$.

(3) Suppose S is E-invariant. Then there is some $C \in \mathscr{H}$ such that for $x \notin C$,

$$m(S, x) = \begin{cases} 1 & x \in S, \\ 0 & x \notin S. \end{cases}$$

(4) If S is E-invariant, then there is some $C \in \mathscr{H}$ such that for $x \in S \setminus C$ we have $m(A, x) = m(A \cap S, x)$.

Proof. (1) This follows from Lemma 1.3.10(1).

(2) This set is E-invariant by (1) and is Δ^1_1 because the local measure functions are Δ^1_1. Now let

$$Y_n = \{x \in Y\colon [A/F_n](x) < [B/F_n](x)\}.$$

The sets Y_n are E-invariant, Δ^1_1 and cover Y, so by Lemma 1.3.10(9) and Lemma 1.3.2(2) we have $A \cap Y \preceq B \cap Y$.

(3) If $x \notin S$ then $[S/F_n](x) = 0$ for all n, so $m(S, x) = 0$. On the other hand, if $x \in S$ then $[S/F_n](x) = k \iff x \in Q_k^{S,F_n}$, so it suffices to show that $\bigcup_{k\neq 2^n} Q_k^{S,F_n} \in \mathscr{H}$. This is done exactly as in the proof of Lemma 1.3.10(7).

(4) Let $C_0(A, B, S)$ be the set constructed in the proof of Lemma 1.3.10(4) and take $C = \bigcup_n C_0(A, F_n, S) \cup C_A \cup C_{A\cap S}$. Then $C \in \mathscr{H}$ by Lemma 1.3.2(2) and clearly C works. ∎

Proposition 1.3.14. *Let $A, B, S \subseteq \mathcal{N}$ be Δ^1_1 and let (A_n) be a uniformly Δ^1_1 sequence of subsets of $\mathcal{N}$.*

(1) If $A \preceq B$ then there is some $C \in \mathscr{H}$ such that $m(A, x) \leq m(B, x)$ for $x \notin C$.

(2) If $A \sim B$ then there is some $C \in \mathscr{H}$ such that $m(A, x) = m(B, x)$ for $x \notin C$.

(3) If A, B are disjoint then there is some $C \in \mathscr{H}$ such that $m(A, x) + m(B, x) = m(A \sqcup B, x)$ for $x \notin C$.

(4) Suppose the (A_n) are pairwise disjoint, S is E-invariant and the partial maps $m(A, \cdot), m(A_n, \cdot)$ are defined on S. Suppose additionally that $m(A, x) > \sum_n m(A_n, x)$ for $x \in S$. Then there is some $C \in \mathscr{H}$ satisfying $(\bigsqcup_n A_n) \cap (S \setminus C) \preceq A \cap (S \setminus C)$.

(5) If $A = \bigsqcup_n A_n$ then there is some $C \in \mathscr{H}$ such that $m(A, x) = \sum_n m(A_n, x)$ for $x \notin C$.

Proof. (1) Let $C = \bigcup_n C_0(A, B, F_n) \cup C_A \cup C_B$, where $C_0(A_0, A_1, B)$ denotes the set constructed in the proof of Lemma 1.3.10(2).

(2) This follows from (1).

(3) Let $C_0(A_0, A_1, B)$ and C_1 be the sets we have constructed in the proofs of Lemma 1.3.10(5) and (7), respectively, and take $C = \bigcup_n C_0(A, B, F_n) \cup C_A \cup C_B \cup C_1$.

(4) We construct recursively a sequence of Δ^1_1 sets and functions $\tilde{A}_n, B_n, C_n, S_n, f_n, g_n$ such that $\tilde{A}_{n+1} = \tilde{A}_n \setminus B_n$, $S_{n+1} = S_n \setminus C_n$, $f_n \colon A_n \cap S_n \sim B_n \cap S_n$, $g_n \colon C_n \prec C_n$, and $m(\tilde{A}_n, x) > \sum_{k\geq n} m(A_k, x)$ for $x \in S_n$. To do this, we first set $\tilde{A}_0 = A$, $S_0 = S$. Now suppose we have $\tilde{A}_n, S_n$ satisfying $m(\tilde{A}_n, x) > \sum_{k\geq n} m(A_k, x)$ for $x \in S_n$. Then $m(\tilde{A}_n, x) > m(A_n, x)$ for $x \in S_n$, so by Lemma 1.3.13(2) we can find $B_n \subseteq \tilde{A}_n$ and $f_n \colon A_n \cap S_n \sim B_n \cap S_n$. By (2), (3) and Lemma 1.3.13(4) there are $g_n \colon C_n \prec C_n$ such that for $x \in S_n \setminus C_n$ we have $m(A_n, x) = m(B_n, x)$ and $m(\tilde{A}_n, x) = m(B_n, x) + m(\tilde{A}_n \setminus B_n, x)$. We then define $\tilde{A}_{n+1} = \tilde{A}_n \setminus B_n$, $S_{n+1} = S_n \setminus C_n$.

By the uniformity of the proofs of (2), (3) and Lemma 1.3.13, these sequences are uniformly Δ^1_1. Let $C = \bigcup_n C_n$, and note that $S \setminus C = \bigcap_n S_n$, so $A_n \cap (S \setminus C) \sim B_n \cap (S \setminus C)$ for all n. Thus by Lemma 1.3.2(2) we have $C \in \mathscr{H}$ and

$$(\bigsqcup_n A_n) \cap (S \setminus C) \sim (\bigsqcup_n B_n) \cap (S \setminus C) \subseteq A \cap (S \setminus C).$$

(5) Let $C_0(A, B), C_1(A, B)$ be the sets constructed in the proofs of (1) and (3), respectively, and let

$$\tilde{C} = C_A \cup \bigcup_n [C_{A_n} \cup C_0(A_0 \cup \cdots \cup A_n, A) \cup C_1(A_0 \cup \cdots \cup A_n, A_{n+1})].$$

Then for $x \notin \tilde{C}$ and $n \in \mathbb{N}$ we have

$$\sum_{k<n} m(A_k, x) = m(\bigcup_{k<n} A_k, x) \leq m(A, x),$$

and therefore $\sum_n m(A_n, x) \leq m(A, x)$ for $x \notin \tilde{C}$.

Now let C_2 be the set constructed in the proof of Lemma 1.3.10(7) and define

$$C = \tilde{C} \cup C_2 \cup C_{\mathcal{N}\setminus A} \cup C_1(A, \mathcal{N} \setminus A) \cup \bigcup_n [C_{F_n} \cup C_{\mathcal{N}\setminus F_n} \cup C_1(F_n, \mathcal{N} \setminus F_n)].$$

Then for $x \notin C$ we have

- $\sum_n m(A_n, x) \leq m(A, x)$,
- $m(A, x) + m(\mathcal{N} \setminus A, x) = m(\mathcal{N}, x)$,
- $\forall n (m(F_n, x) = 2^{-n})$, and
- $\forall n (m(F_n, x) + m(\mathcal{N} \setminus F_n, x) = m(\mathcal{N}, x))$.

Let $S_k = \{x \notin C : m(A, x) > \sum_n m(A_n, x) + 2^{-k}\}$. These sets are Δ^1_1 and E-invariant, and if $x \notin C \cup \bigcup_k S_k$ then $m(A, x) = \sum_n m(A_n, x)$. By the uniformity of the construction of C, S_k and Lemma 1.3.2(2), it remains to show that each $S_k \in \mathscr{H}$.

For $x \in S_k$ we have

$$m(\mathcal{N} \setminus F_k, x) = m(A, x) + m(\mathcal{N} \setminus A, x) - m(F_k, x) > m(\mathcal{N} \setminus A, x) + \sum_n m(A_n, x).$$

By (4) there is some $C_k \in \mathscr{H}$ for which

$$S_k \setminus C_k = \left(\bigcup_n A_n \cup (\mathcal{N} \setminus A) \right) \cap (S_k \setminus C_k) \preceq (\mathcal{N} \setminus F_k) \cap (S_k \setminus C_k).$$

Since F_k is an E-complete section, this means that $S_k \setminus C_k \in \mathscr{H}$, and hence that $S_k \in \mathscr{H}$, as desired. ■

(F) Proof of the Effective Nadkarni's Theorem.

Recall that we have fixed some sequence of maps (γ_n) satisfying (1) of Theorem 1.2.2. Fix now some $\tau, \mathcal{U}, d, (U^k_n)$ satisfying (2), (3) of Theorem 1.2.2. Let C_A be the set defined in Definition 1.3.12, and let $C_0(A, B)$, $C_1(A, B)$, $C_2(A, (A_n))$ be the sets constructed in the proofs of Proposition 1.3.14(2), (3) and (5), respectively. Now define

$$\begin{aligned} C = &\bigcup \{C_U : U \in \mathcal{U}\} \\ &\cup \bigcup \{C_0(U, \gamma_n U) : U \in \mathcal{U}, n \in \mathbb{N}\} \\ &\cup \bigcup \{C_1(U, V \setminus U) : U, V \in \mathcal{U}\} \\ &\cup \bigcup \{C_2(U, (U^k_n)_n) : U \in \mathcal{U}, k > 0\}. \end{aligned}$$

By the uniformity of the constructions of the C_A, C_0, C_1, C_2, along with the fact that $\mathcal{U}, (U_n^k)$ are uniformly Δ_1^1, there is a uniformly Δ_1^1 enumeration of the sets in this union, so C is Δ_1^1. By this uniformity and Lemma 1.3.2(2), C admits a Δ_1^1 compression.

If $\mathcal{N} = C$, then E admits a Δ_1^1 compression. So suppose $\mathcal{N} \neq C$ and fix some $x \in \mathcal{N} \setminus C$. By construction, the following hold for x:

- $m(\emptyset, x) = 0$ and $m(\mathcal{N}, x) = 1$;
- for all $U \in \mathcal{U}$, $m(U, x)$ exists, is zero for $x \notin [U]_E$, and is non-zero and finite for $x \in [U]_E$;
- $m(U, x) = m(\gamma_n U, x)$ for all $U \in \mathcal{U}, n \in \mathbb{N}$;
- $m(U \sqcup V, x) = m(U, x) + m(V, x)$ for all disjoint $U, V \in \mathcal{U}$; and
- for all $U \in \mathcal{U}$ and $k > 0$, $m(U, x) = \sum_n m(U_n^k, x)$.

Now define

$$\mu_x^*(A) = \inf\{\sum_n m(U_n, x) : U_n \in \mathcal{U} \ \ \& \ \ A \subseteq \bigcup_n U_n\}.$$

As in the classical proof of Nadkarni's Theorem (cf. [1, p. 51-52] or [10, Theorem 2.8.1]), μ_x^* is a metric outer measure whose restriction μ_x to the Borel sets is an E-invariant probability Borel measure satisfying $\mu_x(U) = m(U, x)$, for $U \in \mathcal{U}$. Thus, E admits an invariant probability Borel measure.

1.4 A counterexample

Let E be a Δ_1^1 CBER on $\mathcal{N}$. Nadkarni's Theorem says that either E is compressible or E admits an invariant probability Borel measure. We have seen in Theorem 1.1.4 that if E is compressible, then actually there is a Δ_1^1 witness of this. On the other hand, if E is non-compressible, one may ask if there is an effective witness of this, i.e., if E admits a Δ_1^1 invariant probability measure. It turns out that this is true if, for example, E is induced by a continuous, Δ_1^1 action of a countable group on the Cantor space, but it is not true in general.

Let $P(\mathcal{C})$ denote the space of probability Borel measures on $\mathcal{C}$. As with $P(\mathcal{N})$, we identify $P(\mathcal{C})$ with the Π_1^0 set of all $\varphi \in [0,1]^{2^{<\mathbb{N}}}$ satisfying $\varphi(\emptyset) = 1$ and $\varphi(s) = \varphi(s^\frown 0) + \varphi(s^\frown 1)$ for $s \in 2^{<\mathbb{N}}$. We then have the following:

Proposition 1.4.1. *Let E be a CBER on the Cantor space $\mathcal{C}$. Suppose there is a uniformly Δ_1^1 sequence (f_n) of homeomorphisms of $\mathcal{C}$ inducing E, i.e., such that $x E y \iff \exists n(f_n(x) = y)$. Then if E is non-compressible, E admits a Δ_1^1 invariant probability measure.*

Proof. Let $\mathrm{INV}_E \subseteq P(\mathcal{C})$ be the set of all E-invariant probability Borel measures on $\mathcal{C}$. If E is non-compressible, then INV_E is compact, Δ_1^1 and non-empty. By the basis theorem [8, 4F.11], INV_E contains a Δ^1_1 point, which is a Δ^1_1 E-invariant probability measure on C. ∎

Let E, F be CBERs on the standard Borel spaces X, Y respectively. We say that E is **Borel invariantly embeddable** to F, denoted $E \sqsubseteq^i_B F$, if there is an injective Borel map $f : X \to Y$ such that $xEy \iff f(x)Ff(y)$, and such that additionally $f(X) \subseteq Y$ is F-invariant. We say F is **invariantly universal** if $E \sqsubseteq^i_B F$ for any CBER E. Clearly, all invariantly universal CBERs admit invariant probability Borel measures.

Proposition 1.4.2. *There exists an invariantly universal Δ^1_1 CBER on $\mathcal{N}$ that does not admit a Δ^1_1 invariant probability measure.*

Proof. Let $\mathbb{F}_\infty$ be the free group on a countably infinite set of generators, and take F_0 to be the shift equivalence relation on $\mathcal{N}^{\mathbb{F}_\infty} \cong \mathcal{N}$. Note that F_0 is an invariantly universal Δ^1_1 CBER. Let F_1 be a compressible Δ^1_1 CBER on $\mathcal{N}$. Let T be an ill-founded computable tree on $\mathbb{N}$ with no Δ^1_1 branches (cf. [8, 4D.10]), and define the equivalence relation E on $\mathcal{N} \times \mathcal{N}$ by

$$(w, x)E(y, z) \iff w = y \ \ \& \ \ [(w \in [T] \ \ \& \ \ xF_0z) \text{ or } (w \notin [T] \ \ \& \ \ xF_1z)].$$

Then E is a non-compressible invariantly universal Δ^1_1 CBER on $\mathcal{N} \times \mathcal{N} \cong \mathcal{N}$, because T is ill-founded and F_0 is non-compressible and invariantly universal.

Now suppose for the sake of contradiction that E admits a Δ^1_1 invariant probability measure μ. For $s \in \mathbb{N}^{<\mathbb{N}}$, let $N_s = \{x \in \mathcal{N} : s \subseteq x\}$, and define $S = \{s \in \mathbb{N}^{<\mathbb{N}} : \mu(N_s \times \mathcal{N}) > 0\}$. Then S is a non-empty pruned Δ^1_1 subtree of T, because if $s \notin T$ then $E|(N_s \times \mathcal{N})$ is compressible, so $\mu(N_s \times \mathcal{N}) = 0$. But then S, and hence T, has a Δ^1_1 branch, a contradiction. ∎

Remark 1.4.3. Let E be the equivalence relation induced by the shift action of $\mathbb{F}_\infty$ on $C^{\mathbb{F}_\infty}$, and let $Fr(C^{\mathbb{F}_\infty}) \subseteq C^{\mathbb{F}_\infty}$ be the free part of $C^{\mathbb{F}_\infty}$, i.e., the set of points x such that $\gamma x \neq x, \forall \gamma \in \mathbb{F}_\infty, \gamma \neq 1$. Then $E|Fr(C^{\mathbb{F}_\infty})$ is invariantly universal for CBERs that can be induced by a free Borel action of $\mathbb{F}_\infty$.

Using the representation of Δ^1_1 CBERs constructed in Section 1.2, and [8, 4F.14], one sees that the proof of [5, Theorem 3.3.1] is effective. In particular, there is a Δ^1_1, compact, E-invariant set $K \subseteq C^{\mathbb{F}_\infty}$ admitting a Δ^1_1 isomorphism $E|K \cong E|Fr(C^{\mathbb{F}_\infty})$.

Now consider the equivalence relation F on $\mathcal{N} \times C^{\mathbb{F}_\infty}$ given by

$$(w, x)F(y, z) \iff w = y \ \ \& \ \ xEz.$$

Let T be the tree from the proof of Proposition 1.4.2 and let $X = [T] \times Fr(C^{\mathbb{F}_\infty})$. Then $F|X$ is invariantly universal for CBERs that can be induced by a free action of $\mathbb{F}_\infty$, so there is a Borel isomorphism $F|X \cong E|Fr(C^{\mathbb{F}_\infty})$, and $F|X$ does not admit a Δ^1_1 invariant probability Borel measure.

It follows that $F|X$ is Borel isomorphic to a Δ^1_1 compact subshift of $C^{\mathbb{F}_\infty}$. However, by the proof of Proposition 1.4.1, every such subshift admits a Δ^1_1 invariant probability Borel measure, so there is no Δ^1_1 isomorphism between $F|X$ and a Δ^1_1 compact subshift of $C^{\mathbb{F}_\infty}$. In particular, $F|X$ is a concrete witness to [5, Proposition 3.8.15].

1.5 Proof of Effective Ergodic Decomposition

As noted in [9], the proof of Nadkarni's Theorem can be used to provide a proof of the Ergodic Decomposition Theorem (see also [10, Section 2.9]). We will now show that this argument can also be effectivized, providing a proof of the Effective Ergodic Decomposition Theorem for invariant measures from the proof of the Effective Nadkarni's Theorem. This provides a different proof of a special case of Ditzen's Effective Ergodic Decomposition Theorem [2], which is proved more generally for quasi-invariant measures.

Let E be a non-compressible CBER on the Baire space $\mathcal{N}$, in order to prove the Ergodic Decomposition Theorem for E. We may partition $\mathcal{N} = X \sqcup Y$ into Δ^1_1 E-invariant pieces so that $E|X$ is aperiodic and every $E|Y$-class $C \subseteq Y$ is finite. It is easy to see that the Ergodic Decomposition Theorem holds for $E|Y$, so we may assume that E is aperiodic.

Fix $(f_n), \tau, \mathcal{U}, d, (U^k_n)$ satisfying Theorem 1.2.2 for E. By the proof of the Effective Nadkarni's Theorem, there is a Δ^1_1 E-invariant set $C \subseteq \mathcal{N}$ and a local measure function m, such that that C admits a Δ^1_1 compression and for each $x \in \mathcal{N} \setminus C$ there is a (unique) E-invariant probability Borel measure μ_x on X satisfying $\mu_x(U) = m(U, x)$ for all $U \in \mathcal{U}$.

For Δ^1_1 sets $A, B \subseteq \mathcal{N}$, let $Q^{A,B}_n$ be the associated decomposition (cf. Notation 1.3.8). Let F_n be the uniformly Δ^1_1 fundamental sequence for E used in the proof of the Effective Nadkarni's Theorem, and for $s \in \mathbb{N}^{<\mathbb{N}}$, let $N_s = \{x \in \mathcal{N} \colon s \subseteq x\}$. For $s \in \mathbb{N}^{<\mathbb{N}}, n, k \in \mathbb{N}$ define

$$S_{s,n,k} = \begin{cases} (\mathcal{N} \setminus [N_s]_E) \cup Q^{N_s,F_n}_0 & k = 0, \\ Q^{N_s,F_n}_k & \text{otherwise.} \end{cases}$$

By the proof of Theorem 1.2.2, we may assume that $S_{s,n,k} \in \mathcal{U}$ for all s, n, k.

Now let $Z = \mathcal{N} \setminus (C \cup \bigcup_{s,n,k} C_0(S_{s,n,k}))$, where $C_0(S)$ is the set constructed in the proof of Lemma 1.3.13(3). By the uniformity of this construction and Lemma 1.3.2(2), $\mathcal{N} \setminus Z$ is Δ^1_1 and admits a Δ^1_1 compression. By invariance of the local measure function, the assignment $x \mapsto \mu_x$ is E-invariant. Additionally, as noted in the introduction, we may identify $P(\mathcal{N})$ with the subspace of $\varphi \in [0,1]^{\mathbb{N}^{<\mathbb{N}}}$ satisfying $\varphi(\emptyset) = 1$ and $\varphi(s) = \sum_n \varphi(s^\frown n)$, for $s \in \mathbb{N}^{<\mathbb{N}}$. Then, by uniformity in A of the local measure function $m(A, x)$, the assignment $x \mapsto \mu_x$ defines a Δ^1_1 map $Z \to \mathrm{INV}_E \subseteq [0,1]^{\mathbb{N}^{<\mathbb{N}}}$.

For $x \in Z$, let $S_x = \{y \in Z \colon \mu_y = \mu_x\}$.

Lemma 1.5.1. *For any $x \in Z$, $\mu_x(S_x) = 1$.*

Proof. If $x \in S_{s,n,k}$, then by definition of Z, E-invariance of $S_{s,n,k}$ and the fact that $S_{s,n,k} \in \mathcal{U}$, we have $\mu_x(S_{s,n,k}) = m(S_{s,n,k}, x) = 1$.

Now define $\tilde{S}_x = Z \cap \bigcap\{S_{s,n,k} : x \in S_{s,n,k}\}$. Since $\mathcal{N} \setminus Z$ is compressible, $\mu_x(Z) = 1$, and so $\mu_x(\tilde{S}_x) = 1$. If $y \in \tilde{S}_x$, then $[N_s/F_n](x) = [N_s/F_n](y)$ for all s, n, so $\mu_y(N_s) = m(N_s, y) = m(N_s, x) = \mu_x$ for all $s \in \mathbb{N}^{<\mathbb{N}}$, and hence $\mu_y = \mu_x$. Therefore $\tilde{S}_x \subseteq S_x$, and $\mu_x(S_x) = 1$. ■

Lemma 1.5.2. *Let $S \subseteq \mathcal{N}$ be E-invariant and Borel. Then there is an E-invariant compressible Borel set $C \subseteq \mathcal{N}$ such that for $x \notin C$ we have*

$$\mu_x(S) = m(S, x) = \begin{cases} 1 & x \in S, \\ 0 & x \notin S. \end{cases}$$

Proof. By relativizing, we may assume S is Δ^1_1. Repeat the proofs of this section, assuming this time that $S \in \mathcal{U}$, to get a Δ^1_1 set $Z' \subseteq \mathcal{N}$ and a Δ^1_1 assignment $Z' \ni x \mapsto \mu'_x \in \mathrm{INV}_E$ induced by a local measure function m'. Note that $m = m'$ by uniformity of the construction of the local measure function, and hence $\mu_x = \mu'_x$ for $x \in Z \cap Z'$.

Let $C = (\mathcal{N} \setminus Z \cap Z') \cup C_0(S)$, where $C_0(S)$ is the set constructed in the proof of Lemma 1.3.13(3). Then C admits a Δ^1_1 compression, and if $x \notin C$ then

$$\mu_x(S) = \mu'_x(S) = m'(S, x) = \begin{cases} 1 & x \in S, \\ 0 & x \notin S. \end{cases}$$

■

Proposition 1.5.3. *For any $x \in Z$, μ_x is the unique E-ergodic invariant probability Borel measure on $E|S_x$. Moreover, every E-ergodic invariant probability Borel measure is equal to μ_x, for some $x \in Z$.*

Proof. Fix $x \in Z$. Note that S_x is E-invariant, Borel and non-compressible (as it supports the E-invariant measure μ_x). Now let $Y \subseteq \mathcal{N}$ be E-invariant and Borel. By Lemma 1.5.2 there is an E-invariant compressible Borel set $C \subseteq \mathcal{N}$ such that for $y \notin C$, $\mu_y(Y) \in \{0, 1\}$. Since S_x is E-invariant and non-compressible, there must be some $y \in S_x \setminus C$. Then $\mu_x(Y) = \mu_y(Y) \in \{0, 1\}$. Since Y was arbitrary, μ_x is E-ergodic.

Now let ν be any E-ergodic invariant probability Borel measure. For every $s \in \mathbb{N}^{<\mathbb{N}}$, $n \in \mathbb{N}$, there is a unique $k(s, n) \in \mathbb{N}$ such that $\nu(S_{s,n,k(s,n)}) = 1$. Define $S = \bigcap_{s,n} S_{s,n,k(s,n)}$. Then $\nu(S) = 1$, so in particular S is non-compressible, and hence $S \cap Z \neq \emptyset$. Let $x \in S \cap Z$.

We claim that $\mu_x = \nu$. To see this, fix some $s \in \mathbb{N}^{<\mathbb{N}}$, in order to show that $\mu_x(N_s) = \nu(N_s)$. Note that $[N_s/F_n](x) = k(s, n)$, for all s, n, so that $\mu_x(N_s) = \lim_n \frac{k(s,n)}{2^n}$ (cf. Definition 1.3.9 and Definition 1.3.12). We now consider two cases. If $\nu([N_s]_E) = 0$, then $k(s,n)=0$ for all n, so $\mu_x(N_s)=0=\nu(N_s)$. Now suppose $\nu([N_s]_E)=1$. For all n, we have $N_s \cap Q^{N_s,F_n}_{k(s,n)} \approx k(s,n)(F_n \cap Q^{N_s,F_n}_{k(s,n)})$, so, as noted at the start of Section 1.3, $\nu(N_s) \in [k(s,n)2^{-n}, (k(s,n)+1)2^{-n}]$ for all n. Thus

$$\nu(N_s) = \lim_n \frac{k(s,n)}{2^n} = \mu_x(N_s).$$

Finally, it remains to show that μ_x is the unique E-ergodic invariant probability Borel measure on $E|S_x$. To see this, let ν be any other such measure and write $\nu = \mu_y$ for some $y \in Z$. Then $\nu(S_y) = \mu_y(S_y) = 1$, so $\nu(S_x \cap S_y) = 1$. Thus $S_x \cap S_y \neq \emptyset$, and so $\mu_x = \mu_y = \nu$. ■

Proposition 1.5.4. *Let $\mu, \nu \in \mathrm{INV}_E$. If $\mu(S) = \nu(S)$ for all E-invariant Borel sets $S \subseteq \mathcal{N}$, then $\mu = \nu$.*

Proof. Let $A \subseteq \mathcal{N}$ be Δ^1_1. As in the proof of Proposition 1.5.3, we have

$$\mu(A \cap Q^{A,F_n}_k) \in [k2^{-n}\mu(Q^{A,F_n}_k), (k+1)2^{-n}\mu(Q^{A,F_n}_k)].$$

Similarly,

$$\nu(A \cap Q^{A,F_n}_k) \in [k2^{-n}\nu(Q^{A,F_n}_k), (k+1)2^{-n}\nu(Q^{A,F_n}_k)].$$

Since the sets Q^{A,F_n}_k are E-invariant, we have $\mu(Q^{A,F_n}_k) = \nu(Q^{A,F_n}_k)$, and therefore

$$|\mu(A \cap Q^{A,F_n}_k) - \nu(A \cap Q^{A,F_n}_k)| \leq 2^{-n}\mu(Q^{A,F_n}_k).$$

It follows that

$$|\mu(A) - \nu(A)| \leq \sum_k |\mu(A \cap Q^{A,F_n}_k) - \nu(A \cap Q^{A,F_n}_k)| \leq 2^{-n} \sum_k \mu(Q^{A,F_n}_k) \leq 2^{-n}.$$

Since n was arbitrary, $\mu(A) = \nu(A)$. ■

Proposition 1.5.5. *For any $\nu \in \mathrm{INV}_E$, $\nu = \int \mu_x d\nu(x)$.*

Proof. Let $A \subseteq \mathcal{N}$ be E-invariant. Then $\int \mu_x(A) d\nu(x) = \nu(A \cap Z) = \nu(A)$. Thus, by Proposition 1.5.4, $\nu = \int \mu_x d\nu(x)$. ■

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