Alexander S. Kechris

The spaces of measure preserving equivalence relations and graphs

Alexander S. Kechris
Department of Mathematics
California Institute of Technology
Pasadena, CA 91125
kechris@caltech.edu

Contents

1. Introduction

This work is to a large extent a continuation of that in [K] on the global aspects of measure preserving actions of countable groups. We define and study here natural topologies on the spaces of measure preserving equivalence relations and graphs on a standard probability space.

Here is an overview of its contents. In Chapter 2 we recall some basic facts about the space of closed subgroups of a Polish group which admits a (two-sided) invariant metric. In Chapter 3 we discuss full groups of measure preserving countable Borel equivalence relations on a standard probability space (X, μ), including a characterization of these groups among the subgroups of the group $\mathrm{Aut}(X, \mu)$ of measure preserving automorphisms of (X, μ). In Chapter 4 we define a natural Polish topology on the space $S(E)$ of subequivalence relations of a measure preserving countable Borel equivalence relation E. Several formulations of the topology are given and shown to be equivalent. A stronger (non-separable) topology, useful in certain applications, is also discussed. In Chapter 5 we discuss the structure of limits of convergent sequences in $S(E)$. In Chapter 6 it is shown that the topologies on $S(E)$ are coherent under inclusion and can be used to define a topology on the space of all measure preserving countable Borel equivalence relations (which is however far from Polish).

In Chapter 7 we discuss continuity properties of the map that assigns to each measure preserving action of a countable group Γ the associated equivalence relation and show that they are related to properties of the group such as amenability and property (T). In Chapter 8, Chapter 9, and Chapter 10 we include several descriptive set theoretic complexity calculations for classes of equivalence relations in $S(E)$. This leads in Chapter 11 to the introduction and study of the class of richly ergodic equivalence relations E, i.e., those for which the generic equivalence relation in $S(E)$ is ergodic. In Chapter 12 we consider the cost function on the space of subequivalence relations and in Chapter 13 the concept of normality. In Chapter 14 we prove a Borel selection theorem for hyperfiniteness. In Chapter 15 we study the connections of the preceding theory to the structure of invariant, random equivalence relations on a countable group. Chapter 16 deals with ultraproducts of equivalence relations and in Chapter 17 we define and study various notions of factoring for equivalence relations.

In Chapter 18 we introduce an analogous canonical topology on the space $Gr(E)$ of Borel subgraphs of a measure preserving countable Borel equivalence relation E and in Chapter 19 we include various descriptive complexity calculations related to this topology. Chapter 20 deals with the notion of treeability for equivalence relations.

Finally we mention that a survey (without proofs) of the results presented here appeared on [K3].

Acknowledgments. Research in this paper was partially supported by NSF Grants DMS-0968710, DMS-1464475 and DMS-1950475. I would like to thank Lewis Bowen for asking some questions that got me thinking about this topic. Thanks also to Lewis, Peter Burton, François Le Maître, Ben Miller, Forte Shinko , Anush Tserunyan and Robin Tucker-Drob for many useful comments and corrections or for allowing me to include some results of theirs in this paper.

2. The space of closed subgroups

We start with some preliminaries. Fix a Polish metric space (X, d), with $d \leq 1$ and let $\mathcal{F}^*(X)$ be the set of non-empty closed subsets of X. We can identify $F \in \mathcal{F}^*(X)$ with the distance function

$$f_F(x) = d(x, F), x \in X,$$

and view $\mathcal{F}^*(X)$ as a subset of $[0, 1]^X$. The relative topology on $\mathcal{F}^*(X)$ induced by the product topology on $[0, 1]^X$ is called the **Wijsman topology** on $\mathcal{F}^*(X)$. It is the topology generated by the functions:

$$F \mapsto d(x, F), \quad x \in X.$$

It is shown in Beer [B] that $\overline{\mathcal{F}^*(X)}$, the closure of $\mathcal{F}^*(X)$ in $[0, 1]^X$, is compact metrizable and $\mathcal{F}^*(X)$ is G_δ in $\overline{\mathcal{F}^*(X)}$, thus a Polish space. Moreover the Borel σ-algebra of the Wijsman topology on $\mathcal{F}^*(X)$ is the Effros σ-algebra generated by the sets $\{F \in \mathcal{F}^*(X) \colon F \cap V \neq \emptyset\}$, for $V \subseteq X$ open (Hess, see, e.g., [BK, 2.6.2]).

Equivalently, we can describe this topology as follows. Fix a countable dense subset $X_0 \subseteq X$. Then it is clear that this topology is also the one generated by the functions $F \mapsto d(x_0, F), x_0 \in X_0$. Then we can also identify $F \in \mathcal{F}^*(X)$ with the function

$$f_F^0(x_0) = d(x_0, F), x_0 \in X,$$

and view $\mathcal{F}^*(X)$ as a subset of $[0, 1]^{X_0}$. The relative topology on $\mathcal{F}^*(X)$ induced by the product topology on $[0, 1]^{X_0}$ is again the Wijsman topology.

Assume next that Γ is a Polish group with a compatible (two-sided) invariant metric $d \leq 1$. Then (Γ, d) is complete, thus a Polish metric space. Let

$$Sg(\Gamma) = \{H \subseteq \Gamma \colon H \text{ is a closed subgroup}\}.$$

Proposition 2.1. $Sg(\Gamma)$ *is a closed subspace of* $\mathcal{F}^*(\Gamma)$.

Proof. Let $H_n \in Sg(\Gamma)$ and $H_n \to H$. Then $d(1, H_n) = 0 \to d(1, H)$, so $d(1, H) = 0$ and $1 \in H$. Let now $g, h \in H$ in order to show that $gh^{-1} \in H$. Since $0 = d(g, H) = \lim_{n\to\infty} d(g, H_n)$, find $g_n \in H_n$ with $d(g, g_n) \to 0$ and similarly find $h_n \in H_n$ with $d(h, h_n) \to 0$. Then $d(h^{-1}, h_n^{-1}) \to 0$ (since $d(h, h_n) \to 0$ iff $h_n \to h$ iff $h_n^{-1} \to h^{-1}$ iff $d(h^{-1}, h_n^{-1}) \to 0$) and so $d(gh^{-1}, g_n h_n^{-1}) \to 0$, thus $d(gh^{-1}, H_n) \to d(gh^{-1}, H) = 0$, i.e., $gh^{-1} \in H$. □

It follows that $Sg(\Gamma)$ with the induced topology is also Polish. The group Γ acts on $Sg(\Gamma)$ by conjugation: $g \cdot H = gHg^{-1}$.

Proposition 2.2. *The conjugation action of* Γ *on* $Sg(\Gamma)$ *is continuous.*

Proof. It is enough to show that it is separately continuous (see [K2, 9.14]).

(1) Let $g_n \to g$ in Γ and $H \in Sg(\Gamma)$. We will show that $g_n H g_n^{-1} \to gHg^{-1}$, i.e., that for $x \in \Gamma$,

$$d(x, g_n H g_n^{-1}) \to d(x, gHg^{-1}).$$

Now $d(x, g_n H g_n^{-1}) = d(g_n^{-1} x g_n, H)$ and $d(x, gHg^{-1}) = d(g^{-1}xg, H)$. Since $g_n^{-1} x g_n \to g^{-1}xg$ and $|d(x, H) - d(y, H)| \leq d(x, y)$, this is clear.

(2) Let $g \in \Gamma$ and $H_n \to H$ in $Sg(\Gamma)$. We will show that $gH_ng^{-1} \to gHg^{-1}$, i.e., for any $x \in \Gamma$

$$d(x, gH_ng^{-1}) \to d(x, gHg^{-1})$$

or equivalently

$$d(g^{-1}xg, H_n) \to d(g^{-1}xg, H)$$

which is obvious. □

3. Full groups

Let now (X, μ) be a standard probability space (i.e., X is a standard Borel space and μ is a probability Borel measure on X). We denote by $\mathrm{Aut}(X, \mu)$ the group of all Borel automorphisms of X which preserve the measure μ and in which we identify two such automorphisms if they agree μ-a.e.

Unless otherwise stated or is clear from the context, we will assume that (X, μ) is non-atomic and moreover, as usual, we neglect null sets in the sequel.

The **uniform metric** $d = d_u$ on $\mathrm{Aut}(X, \mu)$ is defined by $d_u(S, T) = \mu(\{x\colon S(x) \neq T(x)\})$. This is a (two-sided) invariant complete metric on $\mathrm{Aut}(X, \mu)$ and the associated **uniform topology**, u, makes it a topological group.

Let now E be a measure preserving countable Borel equivalence relation on X and $\Gamma = [E]$ the **full group** of E, i.e., the subgroup of $\mathrm{Aut}(X, \mu)$ consisting of all $T \in \mathrm{Aut}(X, \mu)$ with $T(x)Ex$, for (almost) all x. Then Γ is a closed subgroup of $\mathrm{Aut}(X, \mu)$ in the uniform topology and d restricted to Γ is separable, thus Γ is a Polish group admitting the compatible invariant metric d. When $T \in \mathrm{Aut}(X, \mu)$, its full group $[T]$, is the full group of the equivalence relation E_T induced by T.

For further reference, we also define the **full pseudogroup** of E, in symbols $[[E]]$, consisting of all Borel bijections $f\colon A \to B$, with A, B Borel subsets of X and $f(x)Ex$, for (almost) all $x \in A$, where we again identify any two such functions if they agree a.e. Clearly $[E] \subseteq [[E]]$.

It is an interesting question to characterize the full groups $[E]$ among the subgroups of the topological group $\mathrm{Aut}(X, \mu)$ equipped with the uniform topology, using only the topological group structure of this group. Below we provide one such characterization. We start with the following lemma.

Lemma 3.1. *For any non-trivial involution $T \in \mathrm{Aut}(X,\mu)$, the centralizer C_T of T in the group $\mathrm{Aut}(X,\mu)$ has a largest abelian normal subgroup, denoted by A_T. Moreover $A_T = [T]$.*

Proof. By [K, Lemma 4.7], if $T \in [E]$, E ergodic, then $[T]$ is the largest abelian normal subgroup of $C_T \cap [E]$. Since

$$\mathrm{Aut}(X,\mu) = \bigcup\{[E] \colon E \text{ ergodic}, T \in [E]\}$$

the result follows. □

We now have:

Theorem 3.2. *The following are equivalent for a subgroup Γ of $\mathrm{Aut}(X,\mu)$:*
(i) $\Gamma = [E]$, for a measure preserving countable Borel equivalence relation E,
(ii) (a) Γ is closed and separable in $(\mathrm{Aut}(X,\mu), u)$, (b) every element of Γ is a product of involutions, and (c) for any nontrivial involution $T \in \Gamma$, $A_T \subseteq \Gamma$.

Proof. It is clear that (i) implies (ii), (a) and (c). That (i) implies (ii), (b), see [M1, Chapter 1] or [M2, Theorem 1].

To prove that (ii) implies (i), assume that Γ is nontrivial and let $\mathcal{I}$ be a countable set of nontrivial involutions in Γ which is uniformly dense in the set of all nontrivial involutions in Γ (this is nonempty by (b)). Let E be the equivalence relation induced by $\mathcal{I}$. The group generated by $\mathcal{I}$ is included in $[E]$ and thus so is its uniform closure, so $\Gamma \subseteq [E]$. By [KT, 4.4], the group generated by the union of the full groups $[T]$, $T \in \mathcal{I}$, is uniformly dense in $[E]$. By (c) and Lemma 3.1 this group is contained in Γ, so $\Gamma = [E]$. □

Theorem 3.3. *The following are equivalent for a subgroup Γ of $\mathrm{Aut}(X,\mu)$:*
(i) $\Gamma = [E]$, for an **ergodic** *measure preserving countable Borel equivalence relation E,*
(ii) (a) Γ is closed and separable in $(\mathrm{Aut}(X,\mu), u)$, (b) Γ is simple, and (c) Γ contains a nontrivial involution T with $A_T \subseteq \Gamma$.

Proof. For the proof that (i) implies (ii), use [K, 4.5, 4.6]. For the other direction, it is enough to show that Γ is of the form $[E]$, because (b) then implies the ergodicity of E (see [K, page 22]). By [KT, proof of 4.14], and using (c), we see that there is a nonempty countable set of nontrivial involutions $\mathcal{I}$ contained in Γ such that if E is the equivalence relation induced by $\mathcal{I}$, then $[E] \lhd \Gamma$. Since Γ is simple and $[E]$ is nontrivial, $\Gamma = [E]$. □

Remark 3.4. In Theorem 3.3, (b) can be replaced by (b)*: Γ is topologically simple.

The above characterization of the subgroups Γ of $(\mathrm{Aut}(X,\mu),u)$ that are full groups involves the properties of Γ within $(\mathrm{Aut}(X,\mu),u)$. One can wonder whether there is a characterization that depends only on the topological group structure of (Γ,u). In other words, is it possible to find a property $\mathcal{P}(\Gamma)$ of Polish groups Γ, invariant under topological group isomorphism, such that for a closed separable subgroup of $(\mathrm{Aut}(X,\mu),u)$, Γ is a full group iff $\mathcal{P}(\Gamma)$ holds. It turns out that no such "internal" characterization is possible, even if one uses the metric d_u on Γ.

Proposition 3.5. *For each measure preserving countable Borel equivalence relation E, different than equality, there is a closed separable subgroup G of the group $(\mathrm{Aut}(X,\mu),u)$ which is not a full group but there is an isometry between (G,d_u) and $([E],d_u)$ which is also a group isomorphism.*

Proof. Let $Y = X\times\{0,1\}$, $\nu=\mu\times\eta$, where η is the uniform measure on $\{0,1\}$. Consider the equivalence relation E^* on Y given by

$$(x,i)E^*(y,j) \iff xEy \;\&\; i=j.$$

Then E^* is a measure preserving countable Borel equivalence relation on (Y,ν), which is of course isomorphic to (X,μ). For $T\in[E]$, let $T^*\in[E^*]$ be defined by

$$T^*(x,i) = (T(x),i).$$

Then $T\mapsto T^*$ is an isometry of $([E],d_u)$ with (Γ,d_u), where

$$\Gamma = \{T^*\colon T\in[E]\}$$

is a closed subgroup of $(\mathrm{Aut}(Y,\nu),u)$, and $T\mapsto T^*$ is also clearly a group isomorphism.

However Γ is not a full group. Indeed if $\Gamma=[F]$, then $F=E^*$, so $\Gamma=[E^*]$. But if $T_0\neq T_1$ are in $[E]$, let $S\in[E]$ be defined by

$$S(x,i) = (T_i(x),i).$$

Then $S\in[E^*]\setminus\Gamma$, a contradiction. □

4. The space of subequivalence relations

Let E be a measure preserving countable Borel equivalence relation on (X, μ). We denote by $S(E)$ the set of all subequivalence relations of E, where as usual we identify two such relations if they agree a.e. We will next define a canonical Polish topology on $S(E)$. In fact we will give several equivalent descriptions of this topology.

4.1 The weak topology

We can identify any $F \in S(E)$ with its full group $[F]$ and thus view $S(E)$ as a subspace of $Sg([E])$.

Proposition 4.1. *$S(E)$ is a closed subspace of* $\mathrm{Sg}([E])$.

Proof. Let $F_n \in S(E)$ and $[F_n] \to H \in \mathrm{Sg}([E])$. We will show that $H \in S(E)$, i.e., that $H = [F]$, where F is a subequivalence relation of E.

Let $H_0 \leq H$ be a countable dense subgroup of H. Thus $H_0 \leq [E]$ and let F be the subequivalence relation of E induced by H_0. We will show that this works.

Since $H_0 \subseteq [F]$ and H_0 is dense in H, clearly $H \subseteq [F]$.

We verify next that $[F] \subseteq H$. Fix any $T \in [F]$. Then there is a Borel decomposition $X = \bigsqcup_n A_n$ and $h_n \in H_0$ such that $T = \bigsqcup_n (h_n|A_n)$. Fix $\epsilon > 0$ and let N be so large that

$$\sum_{n \geq N} \mu(A_n) < \epsilon.$$

Put $\varphi = \bigsqcup_{n<N}(h_n|A_n)$. Since $h_0, \ldots, h_{N-1} \in H = \lim_{n\to\infty}[F_n]$, for all large enough N_0 we can find $g_0, \ldots, g_{N-1} \in [F_{N_0}]$ with $d(h_n, g_n) < \frac{\epsilon}{N}$ for $n < N$.

For $n < N$ let $B_n \subseteq A_n$ be such that $\mu(A_n \setminus B_n) < \frac{\epsilon}{N}$ and $h_n|B_n = g_n|B_n$. Let then $\psi = \bigsqcup_{n<N} g_n|B_n$ and note that $\psi(x)F_{N_0}x$ (for almost all x in the domain of ψ). Let $U \in \mathrm{Aut}(X,\mu)$ be such that $\psi \subseteq U$.

We now recall the following fact (see Ioana-Kechris-Tsankov [IKT, 1.1, 1.2]).

Proposition 4.2. *Let F be a measure preserving countable Borel equivalence relation and $S \in \mathrm{Aut}(X,\mu)$. Then*

$$d(S,[F]) = \mu(\{x\colon (x,S(x)) \notin F\})$$

and there is $T \in [F]$ such that $\{x\colon (x,S(x)) \in F\} = \{x\colon S(x) = T(x)\}$, so that, in particular, $d(S,[F]) = d(S,T)$.

By this result, there is $S \in [F_{N_0}]$ such that $S(x) = U(x)$ if $U(x)F_{N_0}x$. Then if $x \in B_n, n < N, U(x) = g_n(x)F_{N_0}x$, so $\psi \subseteq S$.

Now S,T agree except on a set of measure $< N\frac{\epsilon}{N} + \epsilon$, so $d(S,T) < 2\epsilon$ and thus $d(T,[F_{N_0]}) < 2\epsilon$. Thus we have shown that if $T \in [F]$, then we can find $N_1 < N_2 < \dots$ such that $d(T,[F_{N_i}]) < \frac{1}{i}$, so $d(T,H) = 0$, i.e., $T \in H$. □

Below we put

$$d(S,F) = d(S,[F]).$$

From Proposition 4.1, we have that $S(E)$ is also a Polish space with the topology it inherits from $\mathrm{Sg}([E])$. We call this the **weak topology** on $S(E)$ and denote it by w. We recall that in this topology

$$F_n \xrightarrow{w} F \text{ iff } \forall S \in [E](d(S,F_n) \to d(S,F)).$$

Moreover $[E]$ can be replaced in this equivalence by any dense subset of $[E]$. The group $[E]$ acts on $S(E)$ by

$$x(T\cdot F)y \iff T^{-1}(x)FT^{-1}(y)$$

and, since $[T\cdot F] = T[F]T^{-1}$, Proposition 2.2 shows that this action is continuous. It is clear that this action is not minimal, since E is a fixed point of the action. In an earlier version of this work the following problem was raised:

Is there a dense orbit for the action of $[E]$ on $S(E)$?

Subsequently, in [LeM1], Le Maître has shown that the answer is positive when E is ergodic hyperfinite but negative if E is aperiodic, i.e., has infinite classes, and generated by a measure preserving action of an infinite countable group with property (T) (and in fact all non approximable E; see Definition 11.2) . We do not know a characterization of the equivalence relations E for which the above problem has a positive answer.

In fact Le Maître has shown the following, where we use the terminology and notation below:

(a) For each Borel set $A \subseteq X$ and equivalence relation E, we let $E|A = E \cap A^2$ be the **restriction** of E to A.

(b) We say that $F \in S(E)$ has **everywhere infinite index** in E if for every Borel set $A \subseteq X$ of positive measure, $[E|A : F|A] = \infty$, i.e., each $E|A$-class contains infinitely many $F|A$-classes.

(c) id is the equality relation on X.

(d) If $F \in S(E)$, then $[E] \cdot F$ is the orbit of F in the action of $[E]$ on $S(E)$.

(e) For $\mathcal{R} \subseteq S(E)$, $\overline{\mathcal{R}}$ is the closure of $\mathcal{R}$ in the weak topology of $S(E)$.

(f) $\mathcal{H}_E \subseteq S(E)$ is the class of hyperfinite subequivalence relations of E.

Theorem 4.3 (Le Maître, [LeM1]). *(i) If E is aperiodic, then for every $F \in S(E)$, F has everywhere infinite index in E iff $id \in \overline{[E] \cdot F}$.*

(ii) If E is ergodic and $F \in S(E)$ is aperiodic and has everywhere infinite index in E, then $\mathcal{H}_E \subseteq \overline{[E] \cdot F}$. Therefore for $F \in S(E)$, F is aperiodic and has everywhere infinite index in E iff $id \in \overline{[E] \cdot F}$ iff $\mathcal{H}_E \subseteq \overline{[E] \cdot F}$.

In particular, if E is ergodic hyperfinite, then for $F \in S(E)$, F is aperiodic and has everywhere infinite index in E iff $[E] \cdot F$ is dense in $S(E)$.

Moreover Le Maître has shown the following:

Theorem 4.4 (Le Maître, [LeM1]). *If E is ergodic, then all $[E]$ orbits in $\mathcal{H}_E$ are meager in $\mathcal{H}_E$*

In particular, if E is ergodic hyperfinite, then all $[E]$ orbits are meager in $S(E)$

Consider now an aperiodic equivalence relation E and its automorphism group, $N(E)$, which is also the stabilizer of E in $\mathrm{Aut}(X, \mu)$, with its associated Polish topology (see [K, Section 6]). Clearly $N(E)$ also acts on $S(E)$ by the formula:

$$x(T \cdot F)y \iff T^{-1}(x)FT^{-1}(y),$$

or equivalently

$$T \cdot [F] = T[F]T^{-1}.$$

Proposition 4.5. *If E is aperiodic, then the action of $N(E)$ on $S(E)$ is continuous.*

Proof. It is enough to check separate continuity. Below the superscripts on the arrows indicate the spaces in which these limits are taken.

(i) $F_n \xrightarrow{w} F \implies T \cdot F_n \xrightarrow{w} T \cdot F$: This is equivalent to

$$F_n \xrightarrow{w} F \implies d(T^{-1}ST, F_n) \to d(T^{-1}ST, F)$$

for all $S \in [E]$, which is clear as $T^{-1}ST \in [E]$.

(ii) $T_n \xrightarrow{N(E)} T \Rightarrow T_n \cdot F \xrightarrow{w} T \cdot F$: Fix $S \in [E]$ in order to show that

$$d(S, T_n \cdot F) \to d(S, T \cdot F).$$

This is equivalent to

$$d(T_n^{-1}ST_n, F) \to d(T^{-1}ST, F).$$

But $T_n \xrightarrow{N(E)} T$ implies that $T_n^{-1}ST_n \xrightarrow{[E]} T^{-1}ST$, so this is clear. □

Le Maître in [LeM1] has also shown that there is no dense orbit for the action of $N(E)$ on $S(E)$, if E is aperiodic and generated by a measure preserving action of an infinite countable group with property (T).

We finally note, for further reference, the following simple fact:

Proposition 4.6. *Let $F_0 \subseteq F_1 \subseteq \dots$ be in $S(E)$ and let $F = \bigcup_n F_n$. Then $F_n \stackrel{w}{\to} F$. Similarly, if $F_0 \supseteq F_1 \supseteq F_2 \dots$ and $F = \bigcap_n F_n$.*

Proof. This is immediate from Proposition 4.2. □

4.2 The strong topology

We now define another topology on $S(E)$.

Definition 4.7. *Let E be a countable Borel equivalence relation on X. A sequence $(T_i)_{i \in \mathbb{N}}$ of Borel automorphisms is called* **generating** *for E if*

$$T_i(x) E x, \text{ for all } x, i,$$

and if $x \neq y, xEy$, then there is i such that $y = T_i(x)$.

Such generating sequences exist by the Feldman-Moore Theorem (see, e.g., [K4, Section 3.2] and references therein). In fact one can see the following stronger version, where we call a generating sequence $(T_i)_{i\in\mathbb{N}}$ for E a **uniquely generating sequence** if for all $x \neq y, xEy$, there is a unique i such that $y = T_i(x)$. The following is a special case of [KST, 4.10] but we include a proof for completeness.

Proposition 4.8. *Let E be a countable Borel equivalence relation on X. Then there is a uniquely generating sequence $(T_i)_{i\in\mathbb{N}}$ for E consisting of Borel involutions of X.*

Proof. The proof of the Feldman-Moore Theorem gives a sequence $\{S_j\}_{j\in\mathbb{N}}$ of Borel involutions such that $xEy \iff \exists j(S_j(x) = y)$. For each j, let $\mathrm{supp}(S_j) = \{x\colon S_j(x) \neq x\}$. Now let $Y_i = \{x\colon \forall j < i(S_i(x) \neq S_j(x))\}$. If $x \in Y_i$, then $S_i(x) \in Y_i$, since otherwise there is $j < i$ with $x = S_i(S_i(x)) = S_j(S_i(x))$, so $S_j(x) = S_i(x)$, a contradiction. Thus if we let

$$T_i = S_i|Y_i \cup id|(X \setminus Y_i),$$

T_i is an involution and clearly $T_i(x)Ex$. Now let $x \neq y, xEy$. Let i be least such that $S_i(x) = y$. We claim that $x \in Y_i$, thus $T_i(x) = y$. Otherwise, for some $j < i, S_i(x) = S_j(x) = y$ a contradiction. Finally assume that $y = T_i(x) = T_j(x)$ with $j < i$, towards a contradiction. Since $x \neq y$ this means that $x \in \mathrm{supp}(T_j)$, so $y = S_j(x)$, a contradiction. □

Definition 4.9. *Consider the space $S(E)$ and for $T \in [E], F \in S(E)$ let*

$$A_{T,F} = \{x\colon (x, T(x)) \in F\}.$$

Therefore, by Proposition 4.2, $d(T, F) = 1 - \mu(A_{T,F})$. Fix a generating sequence $(T_i)_{i\in\mathbb{N}}$ for E and consider the map

$$F \mapsto (A_{T_i,F})_{i\in\mathbb{N}} \in \mathrm{MALG}^{\mathbb{N}},$$

where $\mathrm{MALG} = \mathrm{MALG}_\mu$ is the measure algebra of (X, μ). We also endow MALG with the usual complete, separable metric

$$d_\mu(A, B) = \mu(A\Delta B),$$

and the associated topology.

Lemma 4.10. *The map $F \mapsto (A_{T_i,F})_{i\in\mathbb{N}}$ is 1-1.*

Proof. Assume $F, G \in S(E)$ and $A_{T_i,F} = A_{T_i,G}$ for all i. Then for $x \neq y$, $(x,y) \in F \iff \exists i(y = T_i(x) \;\&\; x \in A_{T_i,F}) \iff (x,y) \in G$. □

Thus we can identify F with $(A_{T_i,F})_{i\in\mathbb{N}}$ and transfer the product topology on $\mathrm{MALG}^{\mathbb{N}}$ to $S(E)$ to get a separable, metrizable topology on $S(E)$, which we will call the **strong topology** on $S(E)$, Thus

$$F_n \xrightarrow{s} F \iff \forall i(A_{T_i,F_n} \overset{\mathrm{MALG}}{\longrightarrow} A_{T_i,F}).$$

We next note that this topology is independent of the choice of $(T_i)_{i\in\mathbb{N}}$.

Proposition 4.11.

$$F_n \xrightarrow{s} F \iff \forall T \in [E](A_{T,F_n} \overset{\mathrm{MALG}}{\longrightarrow} A_{T,F}).$$

Proof. $\Leftarrow$ is obvious.

$\Rightarrow$ Assume $F_n \xrightarrow{s} F$ and let $T \in [E]$. Let $A_\infty = \{x \colon T(x) = x\}$ and let A_i be a Borel partition of $X \setminus A_\infty$ such that $x \in A_i \Rightarrow T(x) = T_i(x)$, i.e., $X = \bigsqcup_{i=0}^{\infty} A_i \sqcup A_\infty$ and $T|A_i = T_i|A_i$. Fix now $\epsilon > 0$ and choose N large enough so that $\sum_{i\geq N} \mu(A_i) < \epsilon$. Then choose M large enough so that for any $n \geq M$ and any $i < N$,

$$\mu(A_{T_i,F_n} \Delta A_{T_i,F}) < \frac{\epsilon}{N}.$$

Now

$$\begin{aligned} A_{T,F_n} &= \{x \colon (x, T(x)) \in F_n\} \\ &= A_\infty \sqcup \bigsqcup_{i\in\mathbb{N}} (A_i \cap \{x \colon (x,T(x)) \in F_n\}) \\ &= A_\infty \sqcup \bigsqcup_{i<N} (A_i \cap A_{T_i,F_n}) \sqcup C_n, \end{aligned}$$

where $\mu(C_n) < \epsilon$. Similarly

$$\begin{aligned} A_{T,F} &= A_\infty \sqcup \bigsqcup_{i<N} (A_i \cap A_{T,F}) \sqcup C, \\ &= A_\infty \sqcup \bigsqcup_{i<N} (A_i \cap A_{T_i,F}) \sqcup C \end{aligned}$$

where $\mu(C) < \epsilon$. Therefore

$$(A_{T,F_n} \Delta A_{T,F}) \subseteq (\bigsqcup_{i<N} (A_i \cap A_{T_i,F}) \Delta \bigsqcup_{i<N} (A_i \cap A_{T,F_n})) \cup C \cup C_n$$
$$= (\bigsqcup_{i<N} (A_i \cap (A_{T_i,F} \Delta A_{T_i,F_n})) \cup C \cup C_n,$$

thus

$$\mu(A_{T,F_n} \Delta A_{T,F}) \leq \sum_{i<N} \mu(A_{T_i,F} \Delta A_{T_i,F_n}) + 2\epsilon \leq 3\epsilon$$

□

We will next show that the strong topology on $S(E)$ is Polish. Before we do that however we state the following elementary lemma that will be also useful later on. Its proof is straightforward, so we omit it.

Lemma 4.12. *Let Γ be a group, $a \colon \Gamma \times X \to X$ an action of Γ on a set X and put $a(g,x) = g \cdot x$. Let E_a be the induced equivalence relation on X and let $F \subseteq E_a$ be a subequivalence relation. For $g \in \Gamma$, let*

$$A^a_{g,F} = A_{g,F} = \{x \colon (x, g \cdot x) \in F\}.$$

Then for all $g, h \in \Gamma$,

1. $A_{1,F} = X$,
2. $A_{g,F} \subseteq g^{-1} \cdot A_{g^{-1},F}$,
3. $A_{g,F} \cap g^{-1} \cdot A_{h,F} \subseteq A_{hg,F}$,
4. $A_{h,F} \cap \mathrm{Fix}(h^{-1}g) \subseteq A_{g,F}$,

where

$$\mathrm{Fix}(p) = \{x \colon p \cdot x = x\}.$$

Conversely, if $(A_g)_{g\in\Gamma}$ is a family of sets satisfying 1.-3. above, then the relation

$$xFy \iff \exists g (g \cdot x = y \vee x \in A_g)$$

defines a subequivalence relation of E_a and if 4. also holds we have that $A_g = A_{g,F}$, for all $g \in \Gamma$.

Theorem 4.13. *The strong topology on $S(E)$ is Polish.*

Proof. Proposition 4.11 shows that the strong topology does not depend on which generating sequence we use. So fix a Borel action of a countable group Γ generating E and for each group element g denote also by g the automorphism of the space X induced by the action of g. Since the strong topology is obtained by transferring to $S(E)$ the relative topology (in MALG^Γ) of the range of the map

$$F \mapsto (A_{g,F})_{g\in\Gamma},$$

it is enough to show that the range of this map is closed in MALG^Γ. This means that we have to show that if $F_n \in S(E)$ and for each g, $A_{g,F_n} \stackrel{\mathrm{MALG}}{\longrightarrow} A_g$ as $n \to \infty$, then there is $F \in S(E)$ with $A_{g,F} = A_g$, for all $g \in \Gamma$.

Since, for each n, the family $(A_{g,F_n})_{g\in\Gamma}$ satisfies (a.e.) conditions 1.-4. of Lemma 4.12, it follows, by taking limits, that so does the family $(A_g)_{g\in\Gamma}$, and then, by Lemma 4.12 again, there is $F \in S(E)$ such that $A_{g,F} = A_g$, for all $g \in \Gamma$. □

An alternative description of the strong topology on $S(E)$ is as follows:

First consider MALG and let $\mathcal{D} \subseteq \mathrm{MALG}$ be a countable dense subset of MALG. Then the map

$$A \in \mathrm{MALG} \mapsto \{\mu(A \cap D)\}_{D\in\mathcal{D}} \in [0,1]^{\mathcal{D}}$$

is 1-1. Because if $A, B \in \mathrm{MALG}$ are distinct, then either $\mu(A \setminus B) > 0$ or $\mu(B\setminus A) > 0$. Say $a = \mu(A\setminus B) > 0$. Let $D \in \mathcal{D}$ be such $\mu((A\setminus B)\Delta D) < a/2$. Then $\mu(D \cap A) \geq \mu(D \cap (A \setminus B)) = a - \mu((A \setminus B) \setminus D) > a/2$, while $\mu(D \cap B) \leq \mu((D \cap (X \setminus A)) \cup (D \cap B)) = \mu(D \setminus (A \setminus B)) < a/2$, so $\mu(A \cap D) \neq \mu(B \cap D)$. Thus we can identify MALG with the range of this map and transfer to MALG the relative topology from $[0,1]^{\mathcal{D}}$ (with the product topology). We can now see that this topology is the same as the topology of MALG. This follows from the next proposition.

Proposition 4.14. *The following are equivalent:*

(i) $A_n \stackrel{\mathrm{MALG}}{\longrightarrow} A$,

(ii) $\forall B \in \mathrm{MALG}(\mu(B \cap A_n) \to \mu(B \cap A))$,

(iii) $\forall D \in \mathcal{D}(\mu(D \cap A_n) \to \mu(D \cap A))$.

Proof. Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). To see that (ii) $\Rightarrow$ (i), take in (ii) $B = A$ and $B = X \setminus A$. Finally for (iii) $\Rightarrow$ (ii), fix $B \in \mathrm{MALG}$ and $\epsilon > 0$. Then let $D \in \mathcal{D}$ be such that $\mu(D \Delta B) < \epsilon$. Choose next N such that for $n \geq N$ we have $|\mu(D \cap A_n) - \mu(D \cap A)| \leq \epsilon$. Then for such n, $|\mu(B \cap A_n) - \mu(B \cap A)| \leq |\mu(B \cap A_n) - \mu(D \cap A_n)| + |\mu(D \cap A_n) - \mu(D \cap A)| + |\mu(D \cap A) - \mu(B \cap A)| < 2\mu(B \Delta D) + \epsilon < 3\epsilon$. $\square$

From this it follows that if $(T_i)_{i \in \mathbb{N}}$ is a generating sequence for E and $\mathcal{D}$ is a dense subset of MALG,

$$\begin{aligned} F_n \xrightarrow{s} F &\iff \forall i \in \mathbb{N} \forall D \in \mathcal{D}(\mu(A_{T_i,F_n} \cap D) \to \mu(A_{T_i,F} \cap D)) \\ &\iff \forall T \in [E] \forall D \in \mathcal{D}(\mu(A_{T,F_n} \cap D) \to \mu(A_{T,F} \cap D)) \\ &\iff \forall T \in [E] \forall B \in \mathrm{MALG}(\mu(A_{T,F_n} \cap B) \to \mu(A_{T,F} \cap B)). \end{aligned}$$

For comparison we note that

$$F_n \xrightarrow{w} F \iff \forall T \in [E](\mu(A_{T,F_n}) \to \mu(A_{T,F})).$$

Moreover in all these equivalences we can replace $[E]$ by any dense subset of $[E]$.

4.3 Identification of the topologies

We next show that the two topologies we introduced are the same.

Theorem 4.15. *The weak topology on $S(E)$ is equal to the strong topology on $S(E)$.*

Proof. Clearly the weak topology is contained in the strong topology, so it is enough to show that if $F_n, F \in S(E)$ and $F_n \xrightarrow{w} F$, then $F_n \xrightarrow{s} F$.

So assume that $F_n \xrightarrow{w} F$. Fix $T \in [E]$ in order to show that $\mu(A_{T,F_n} \triangle A_{T,F}) \to 0$. By Proposition 4.2, let $S \in [F]$ be such that $(x, T(x)) \in F \iff S(x) = T(x)$. The rest of the argument is due to Anush Tserunyan. My original proof was more complicated.

We have $\mu(A_{T,F} \setminus A_{T,F_n}) = \mu(\{x \in A_{T,F} \colon (x, T(x)) \notin F_n\}) = \mu(\{x \in A_{T,F} \colon (x, S(x)) \notin F_n\})\} \leq d(S, F_n) \to 0$.

Also $\mu(A_{T,F_n} \setminus A_{T,F}) - \mu(A_{T,F} \setminus A_{T,F_n}) = (\mu(A_{T,F_n} \setminus A_{T,F}) + \mu(A_{T,F_n} \cap A_{T,F})) - (\mu(A_{T,F} \cap A_{T,F_n}) + \mu(A_{T,F} \setminus A_{T,F_n})) = \mu(A_{T,F_n}) - \mu(A_{T,F}) \to 0$, and hence $\mu(A_{T,F_n} \setminus A_{T,F}) \to 0$, so $\mu(A_{T,F_n} \triangle A_{T,F}) \to 0$. $\square$

Remark 4.16 (A. Tserunyan)**.** Note that in the proof of Theorem 4.15 we only needed to verify that $F_n \xrightarrow{w} F \Rightarrow \forall i(\mu(A_{T_i,F_n} \bigtriangleup A_{T_i,F}) \to 0)$, for a sequence of involutions $(T_i)_{i\in\mathbb{N}}$ generating E. For an involution T, it is obvious how to find $S \in [F]$ such that $S(x) = T(x)$, whenever $(x, T(x)) \in F$. One simply defines $A = \{x : (x, T(x)) \in F\}$ and, noting that A is T-invariant, let $S(x) = T(x)$, if $x \in A$, and $S(x) = x$, if $x \notin A$.

From now on we will call this topology simply the **topology** of $S(E)$. Note that we also have the following characterization of convergence in this topology.

Let $((E))$ be the set of Borel maps $\varphi\colon A \to B$, with A, B Borel subsets of X, such that $x \in A \Rightarrow \varphi(x)Ex$. For $\varphi \in ((E)), F \in S(E)$, let

$$A_{\varphi,F} = \{x \in \mathrm{dom}(\varphi)\colon (x, \varphi(x)) \in F\}.$$

Then for $F_n, F \in S(E)$:

$$\begin{aligned} F_n \to F &\iff \forall\varphi \in ((E))(\mu(A_{\varphi,F_n}) \to \mu(A_{\varphi,F})) \\ &\iff \forall\varphi \in ((E))(A_{\varphi,F_n} \xrightarrow{\mathrm{MALG}} A_{\varphi,F}). \end{aligned}$$

This is because if $\varphi \in ((E))$ and (T_i) is such that $xEy \iff \exists i(y = T_i(x))$, then there is a Borel decomposition $\mathrm{dom}(\varphi) = \bigsqcup_i A_i$ such that $x \in A_i \Rightarrow \varphi(x) = T_i(x)$. Then if $F_n \to F$, we have $\mu(A_{\varphi,F_n} \bigtriangleup A_{\varphi,F}) = \sum_i \mu(A_i \cap (A_{T_i,F_n} \bigtriangleup A_{T_i,F})) \to 0$, as $n \to \infty$.

Remark 4.17. One can also consider the topology on $S(E)$ induced by the complete metric

$$\sigma(F_1, F_2) = \mu(\{x\colon [x]_{F_1} \neq [x]_{F_2}\})$$

(see [CM1, 1.7]). This is stronger than the topology of $S(E)$ but it is not separable in general. However it is shown in [CM1, Proposition 1.7.4] that it is separable when restricted to the finite subequivalence relations of E.

4.4 Alternative descriptions

We discuss here three more equivalent descriptions of the topology of $S(E)$.

(1) If Y is a standard Borel space and ν, ρ are Borel probability measures on Y that are *equivalent*, i.e., have the same null sets, then $\mathrm{MALG}_\nu =$

MALG_ρ. The measure ν induces the usual Polish metric $\delta_\nu(A,B) = \nu(A\Delta B)$ on MALG_ν and similarly for ρ. Since ν is equivalent to ρ, these two metrics are equivalent, i.e., induce the same topology. In particular, if Σ is a σ-finite Borel measure on Y one can define unambiguously a Polish topology on $\mathrm{MALG}_\Sigma = \mathrm{MALG}_\nu$, for any Borel probability measure ν equivalent to Σ, e.g., $\nu = \sum_{n=0}^\infty \frac{1}{2^n}(\Sigma_{Y_n})$, where $Y = \bigsqcup_n Y_n$ is a Borel decomposition of Y into sets of positive finite Σ-measure and Σ_{Y_n} is the normalized restriction of Σ to Y_n.

The set $E \subseteq X^2$ admits a Borel measure $M = M_E$ defined by

$$M(W) = \int |W_x| d\mu(x) = \int |W^y| d\mu(y)$$

for Borel $W \subseteq E$, where $W_x = \{y\colon (x,y) \in W\}, W^y = \{x\colon (x,y)\in W\}$. This measure is σ-finite. We call the measure algebra of M, the *measure algebra of* E, in symbols MALG_E. (Thus $\mathrm{MALG}_E = \mathrm{MALG}_M$.) Fix a sequence $(T_i)_{i\in\mathbb{N}}$ in $[E]$, such that $xEy \iff \exists i(T_i(x) = y)$. Note that $M(\mathrm{graph}(T_i)) = 1$. Define next the Borel probability measure $\nu = \nu_{(T_i)}$ on E by

$$\begin{aligned}\nu(W) &= \sum_{i=0}^\infty \frac{1}{2^{i+1}} M(W \cap \mathrm{graph}(T_i)).\\ &= \sum_{i=0}^\infty \frac{1}{2^{i+1}} \mu(A_{T_i,W})\end{aligned}$$

where for $T \in [E], W \subseteq E$:

$$A_{T,W} = \{x\colon (x, T(x)) \in W\}.$$

This is equivalent to M and the metric

$$\delta_\nu(W,V) = \nu(W\Delta V).$$

gives the topology of $\mathrm{MALG}_E = \mathrm{MALG}_\nu$. In this topology

$$W_n \to W \iff \forall i(A_{T_i,W_n} \stackrel{\mathrm{MALG}}{\longrightarrow} A_{T_i,W}).$$

It is clear that $S(E) \subseteq \mathrm{MALG}_E$ and the topology of $S(E)$ is the induced topology from MALG_E. Also as in the proof of Theorem 4.13, $S(E)$ is a closed set in the topology of MALG_E. Thus we can view $S(E)$ as a closed subspace of MALG_E.

We next note a selection property of this representation of $S(E)$. Below we view E as a genuine countable Borel equivalence relation on X and not one viewed μ-a.e. The measure ν as above is a non-atomic probability measure on E and therefore there is a Borel bijection $\theta\colon E \to (0,1)$, which takes ν to the Lebesgue measure λ on $(0,1)$. For any $A \in \mathrm{MALG}_\lambda$, let $\varphi(A) = \{x \in (0,1)\colon x$ is a density point of $A\}$. Then $\varphi(A)$ is a Borel subset of $(0,1)$ which represents A in the measure algebra MALG_λ. So for each $F \in S(E)$, let

$$\psi(F) = \theta^{-1}(\varphi(\theta(F)).$$

Then $\psi(F)$ represents F in the measure algebra MALG_ν. Let $\tilde{F}$ be a Borel subequivalence relation of E that represents F in the measure algebra MALG_ν. Then $\nu(\tilde{F}\Delta\psi(F)) = 0$, so there is a Borel E-invariant set $L \subseteq X$ with $\mu(L) = 1$ and for $x, y \in L, (x,y) \in \tilde{F} \iff (x,y) \in \psi(F)$. Let

$$B = \{x \in X \colon \psi(F)|[x]_E \text{ is an equivalence relation}\}.$$

Then $\mu(B) = 1$ and B is E-invariant Borel. Let $F^\circ = (\psi(F)|B) \cup \{(x,x)\colon x \notin B\}$. Then F° is a Borel subequivalence relation of E and F° represents F in MALG_ν. In fact a simple calculation shows that F° has a uniform Borel definition from F, i.e., we have the following:

Proposition 4.18. *There is a Borel set $R \subseteq S(E) \times E$ such that for any $F \in S(E)$, the section $R_F = F^\circ$ is a Borel subequivalence relation of E which represents F in the measure algebra MALG_ν, i.e., F° is equal to F in $S(E)$.*

We can also use this result to formulate a "uniform Borel version" of the Ergodic Decomposition Theorem for elements of $S(E)$.

First recall the **Ergodic Decomposition Theorem** of Farrell and (independently) Varadarajan, where again below F is viewed as a genuine countable Borel equivalence relation on X.

Theorem 4.19 (Farrell [F], Varadarajan [V]). *Let F be a countable Borel equivalence relation on a standard Borel space X. Then*

$$\mathcal{EI}_F = \{\sigma \in P(X) \colon \sigma \textit{ is invariant, ergodic for } F\}$$

is a Borel set in the standard Borel space $P(X)$ of probability measures on X and if F admits an invariant probability Borel measure, then $\mathcal{EI}_F \neq \emptyset$, and there is a Borel surjection $\pi : X \to \mathcal{EI}_F$ such that

1. *π is F-invariant,*

2. *if $X_e = \{x : \pi(x) = e\}$, then $e(X_e) = 1$ and $F|X_e$ has a unique invariant measure, namely e,*

3. *if $\mu \in P(X)$ is invariant for F, then $\mu = \int \pi(x)\, d\mu(x)$.*

Moreover, π is uniquely determined in the sense that, if π' is another such map, then $\{x : \pi(x) \neq \pi'(x)\}$ is null with respect to all invariant measures for F.

The proof of this result is effective and therefore, in combination with Proposition 4.18, shows the following:

Theorem 4.20. *Let $E, R, F \mapsto F^\circ$ be as in Proposition 4.18. Let $Q \subseteq S(E) \times P(X)$ be defined by*

$$(F, \sigma) \in Q \iff \sigma \in \mathcal{EI}_{F^\circ}.$$

Then Q is Borel, nonempty and there is a Borel set $R \subseteq S(E) \times X \times P(X)$ such that for each $F \in S(E)$, the section $R_F \subseteq X \times P(X)$ is the graph of a (Borel) function π_F which is an ergodic decomposition for F° as in Theorem 4.19.

(2) The next description is due to Robin Tucker-Drob and the author. It is motivated by the idea of **measurable subgroups**, see [Bo, Section 4].

First, without loss of generality, we can assume that $X = 2^{\mathbb{N}}$ and E is generated by a continuous action of a countable group Γ. (Recall here that we identify equivalence relations if they agree a.e.)

For $x \in X, F \in S(E)$, define $\Gamma_x^F = \{g \in \Gamma\colon (x, g^{-1} \cdot x) \in F\}$. Then $\Gamma_x^F \in \mathcal{P}_1(\Gamma) = \{a \subseteq \Gamma\colon 1 \in a\}$. For $g \in \Gamma, a \in \mathcal{P}_1(\Gamma)$, let $ga = \{gh\colon h \in a\}$ and $a^{-1} = \{h^{-1}\colon h \in a\}$. Put

$$\varphi_F(x) = (x, \Gamma_x^F) \in X \times \mathcal{P}_1(\Gamma).$$

On $X \times \mathcal{P}_1(\Gamma)$ put

$$(x, a) R (y, b) \iff \exists g \in a^{-1} (g \cdot x = y \,\&\, ga = b).$$

Proposition 4.21. *R is an equivalence relation.*

Proof. This is obvious. □

Proposition 4.22. *$\varphi_F\colon X \to X \times \mathcal{P}_1(\Gamma)$ is 1-1.*

Proof. This is also obvious. □

Proposition 4.23. *$\varphi_F(X)$ is R-invariant.*

Proof. Let $(x,\Gamma_x^F)R(y,b)$ and let $g \in \Gamma$ be such that $g \cdot x = y, g\Gamma_x^F = b$. So $(x, g \cdot x) \in F$, thus $(x,y) \in F$. Now $\Gamma_y^F = \{h\colon (y, h^{-1} \cdot y) \in F\} = \{h\colon (y, h^{-1}g \cdot x) \in F\} = \{h\colon (x, h^{-1}g \cdot x) \in F\} = \{gp\colon (x, p^{-1} \cdot x) \in F\} = g\{p\colon (x, p^{-1} \cdot x) \in F\} = g\Gamma_x^F = b$, so $(y,b) = \varphi_F(y)$. □

Proposition 4.24. $xFy \iff \varphi_F(x)R\varphi_F(y)$.

Proof. $\Rightarrow$ Let xFy and let $g \in \Gamma$ be such that $y = g \cdot x$. That $g\Gamma_x^F = \Gamma_y^F$ follows as in Proposition 4.23.

$\Leftarrow$ Let $g \in \Gamma$ be such that $g \cdot x = y$ and $g\Gamma_x^F = \Gamma_y^F$. Then $g^{-1} \in \Gamma_x^F$, so $(x, g \cdot x) = (x,y) \in F$. □

Since μ is F-invariant and φ_F is a Borel bijection between X and a Borel R-invariant subset of $X \times \mathcal{P}_1(\Gamma)$, it follows that $(\varphi_F)_*\mu = \mu_F$ is an R-invariant probability measure on $X \times \mathcal{P}_1(\Gamma)$.

Remark 4.25. Actually the definition of $\varphi_F, F \in S(E)$, depends on picking an a.e. representative for F but it is easy to check that μ_F is well defined.

Let $\mathcal{M}$ be the compact, metrizable space of probability measures on the compact zero-dimensional space $Y = X \times \mathcal{P}_1(\Gamma) \subseteq X \times \mathcal{P}(\Gamma)$, where $\mathcal{P}(\Gamma) = \{a\colon a \subseteq \Gamma\}$ (we identify of course here $\mathcal{P}(\Gamma)$ with the product space 2^Γ). We first note the following:

Proposition 4.26. *$\{\mu \in \mathcal{M}\colon \mu$ is R-invariant$\}$ is closed in $\mathcal{M}$.*

Proof. For $g \in \Gamma$, let $N_g = \{(x,a) \in Y\colon g \in a^{-1}\}$, a clopen subset of Y. Let Γ act on $X \times \mathcal{P}(\Gamma)$ by $g \cdot (x.a) = (g \cdot x, ga)$. Of course Y is not invariant under this action but note that $g \cdot N_g \subseteq Y$. It is enough to show for $\mu \in \mathcal{M}$ the following:

Claim. For $\mu \in \mathcal{M}$, μ is R-invariant $\iff \forall g \forall$ clopen $N \subseteq N_g(\mu(N) = \mu(g \cdot N))$.

Granting this claim, it is clear that $\{\mu \in \mathcal{M}\colon \mu$ is R-invariant$\}$ is closed in $\mathcal{M}$.

Proof of the claim. $\Rightarrow$ Fix $N \subseteq N_g, N$ clopen. Then $N \ni t \mapsto g \cdot t$ is in $[[R]]$, so $\mu(N) = \mu(g \cdot N)$ follows.

$\Leftarrow$ Let $\varphi\colon A \to B$ be in $[[R]]$, in order to show that $\mu(A) = \mu(\varphi(A))$. Now $\varphi = \bigsqcup_{g\in\Gamma} \varphi_g, \varphi_g\colon A_g \to B_g, A = \bigsqcup_{g\in\Gamma} A_g, B = \bigsqcup_{g\in\Gamma} B_g, A_g, B_g$ Borel and $A_g \subseteq N_g, \varphi_g(y) = g \cdot y$ for $y \in A_g$. It is thus enough to show that $\mu(B_g) = \mu(A_g)$. Since $A_g \subseteq N_g$, it is enough to show that for every Borel set $A \subseteq N_g, \mu(A) = \mu(g \cdot A)$. Let $\mathcal{B} = \{A \subseteq N_g\colon A \text{ is Borel and } \mu(A) = \mu(g \cdot A)\}$. By hypothesis $\mathcal{B}$ contains the algebra of clopen sets contained in (the clopen set) N_g and clearly $\mathcal{B}$ is closed under relative complementation in N_g and under countable disjoint unions, so $\mathcal{B}$ contains all the Borel subsets of N_g. □

Define now $\Phi\colon S(E) \to \mathcal{M}$ by $\Phi(F) = \mu_F = (\varphi_F)_*\mu$.

Theorem 4.27. *The map $\Phi\colon S(E) \to \mathcal{M}$ is a homeomorphism of $S(E)$ with a (necessarily) G_δ subspace of $\mathcal{M}$.*

Proof. (a) Φ is continuous: It is enough to show that for each clopen rectangle $U \times V$ in $Y = X \times P_1(\Gamma)$, the function

$$F \in S(E) \mapsto (\varphi_F)_*\mu(U \times V)$$

is continuous. Now V is a finite disjoint union of sets of the form

$$W = \{a \in \mathcal{P}_1(\Gamma)\colon g_1^{-1} \in a \ \& \ \dots \ \& \ g_n^{-1} \in a \ \& \ h_1^{-1} \notin a \ \& \dots \& \ h_m^{-1} \notin a\}$$

for $g_i, h_j \in \Gamma$, so it is enough to show that

$$F \in S(E) \mapsto (\varphi_F)_*\mu(U \times W)$$

is continuous. We have

$$\begin{aligned}
(\varphi_F)_*\mu(U \times W) &= \mu(\varphi_F^{-1}(U \times W)) \\
&= \mu(\{x\colon x \in U \ \& \ g_1^{-1} \in \Gamma_x^F \ \& \dots \& \ g_n^{-1} \in \Gamma_x^F \\
&\qquad \& \ h_1^{-1} \notin \Gamma_x^F \ \& \dots \& \ h_m^{-1} \notin \Gamma_x^F\} \\
&= \mu(\{x\colon x \in U \ \& \ x \in A_{g_1,F} \ \& \dots \& \ x \in A_{g_n,F} \\
&\qquad \& \ x \notin A_{h_1,F} \ \& \dots \& \ x \notin A_{h_m,F}\}) \\
&= \mu(U \cap A_{g_1,F} \cap \dots \cap A_{g_n,F} \\
&\qquad \cap (X \setminus A_{h_1,F}) \cap \dots \cap (X \setminus A_{h_m,F})].
\end{aligned}$$

This function is continuous in the (strong) topology of $S(E)$, so Φ is continuous.

(b) Φ is 1-1: For Borel $B \subseteq X, g \in \Gamma$, let $N_{g,B} = B \times \{a \in \mathcal{P}_1(\Gamma) : g^{-1} \in a\}$. Then $\mu_F(N_{g,B}) = \mu(B \cap A_{g,F})$. Thus $\Phi(F) = \Phi(F')$ implies that $\mu(B \cap A_{g,F}) = \mu(B \cap A_{g,F'})$, for any B, g, so $A_{g,F} = A_{g,F'}$ and $F = F'$.

(c) Φ^{-1} is continuous: We check that

$$\mu_{F_n} \to \mu_F \Rightarrow F_n \to F.$$

Let $\mathcal{D}$ be the collection of clopen sets in X, so that $\mathcal{D} \subseteq \mathrm{MALG}$ is countable dense in MALG. Then for $D \in \mathcal{D}$, $N_{g,D}$ is clopen in $X \times \mathcal{P}_1(\Gamma)$, so $\mu_{F_n}(N_{g,D}) \to \mu_F(N_{g,D})$. Thus $\mu(D \cap A_{g,F_n}) \to \mu(D \cap A_{g,F})$ for all $D \in \mathcal{D}, g \in \Gamma$, therefore, by Proposition 4.14, $A_{g,F_n} \overset{\mathrm{MALG}}{\longrightarrow} A_{g,F}$ for all $g \in \Gamma$, and so $F_n \to F$. □

Thus the topological space $S(E)$ can be identified with a G_δ subspace of $\mathcal{M}$ and this gives another description of the topology of $S(E)$.

(3) The final description is due to Peter Burton.

Let Γ be a countable group and let $A(\Gamma, X, \mu)$ be the space of measure preserving actions of Γ on (X, μ). Denote by $(A(\Gamma, X, \mu), u)$ the space of measure preserving actions of Γ on (X, μ) with the **uniform topology** u (see [K, Section 10, **(A)**]). Here we consider the product topology on $\mathrm{Aut}(X, \mu)^\Gamma$, where $\mathrm{Aut}(X, \mu)$ is given the uniform topology. The space $A(\Gamma, X, \mu)$ is then viewed as a closed subspace of $\mathrm{Aut}(X, \mu)^\Gamma$ in this product topology. Given an equivalence relation E, we denote by $A(\Gamma, E) = A(\Gamma, [E])$ the subspace of $A(\Gamma, X, \mu)$ consisting of all $a \in A(\Gamma, X, \mu)$ "contained" in E, i.e., $\forall \gamma \in \Gamma(\gamma^a \in [E])$ (see [K1, Section 6]). Then $A(\Gamma, E)$ is separable and closed in $(A(\Gamma, X, \mu), u)$, so a Polish space in the uniform topology.

Consider now the case $\Gamma = \mathbb{F}_\infty$, the free group with a countably infinite sequence of free generators (γ_i). Then a complete compatible metric for $(A(\Gamma, E), u)$ is given by

$$\delta(a_1, a_2) = \sum_{i=0}^{\infty} 2^{-(i+1)} d_u(\gamma_i^{a_1}, \gamma_i^{a_2}).$$

Fix a generating sequence of involutions (T_i) for E. Then the following is a compatible metric for the topology of $S(E)$

$$\rho(F_1, F_2) = \sum_{i=0}^{\infty} 2^{-(i+1)} \mu(A_{T_i,F_1} \Delta A_{T_i,F_2})$$

(see Section 4.2).

Note that the metric ρ is complete. Indeed if (F_n) is a ρ-Cauchy sequence, then for each i, $(A_{T_i,F_n})_n$ is a Cauchy sequence in the usual metric of MALG given by $\mu(A\Delta B)$. The argument in the proof of Proposition 4.11 shows then that for each $T \in [E]$ the sequence $(A_{T,F_n})_n$ is Cauchy in the metric of MALG and thus converges to some A_T. Then the argument in the proof of Theorem 4.13 shows that there is an $F \in S(E)$ such that $F_n \to F$.

We define a map $\Psi\colon S(E) \to A(\mathbb{F}_\infty, E)$ as follows: We let $\Psi(F) = a$, where the action a is defined by letting $\gamma_i^a(x) = T_i(x)$, if $T_i(x)Fx$, and $\gamma_i^a(x) = x$, otherwise. Denoting by E_a the equivalence relation generated by an action a, we have that $F = E_{\Psi(F)}$.

Theorem 4.28. Ψ *is an isometric embedding of* $(S(E), \rho)$ *onto a closed subspace of* $(A(\mathbb{F}_\infty, E), \delta)$.

Proof. To show that Ψ is an isometry it is enough to check that for each i, and each F_1, F_2 in $S(E)$ with $\Psi(F_1) = a_1, \Psi(F_2) = a_2$, we have

$$\{x\colon \gamma_i^{a_1}(x) \neq \gamma_i^{a_2}(x)\} = A_{T_i,F_1} \Delta A_{T_i,F_2}.$$

which follows easily from the definitions. Finally the range of Ψ is closed, since the metric δ is complete. $\square$

Therefore the topological space $S(E)$ can be identified with a closed subspace of $(A(\mathbb{F}_\infty, E), u)$.

4.5 Continuity of operations

We discuss here the continuity (or lack thereof) of various operations in $S(E)$.

The operation $(F_1, F_2) \mapsto F_1 \cap F_2$ from $S(E) \times S(E)$ to $S(E)$ is continuous. The relations $F_1 \subseteq F_2$ and $F_1 \perp F_2$ (see [KM, Section 27] are closed in $S(E) \times S(E)$. Moreover the map $(F_1, F_2) \mapsto F_1 \times F_2$ from $S(E_1) \times S(E_2)$ to $S(E_1 \times E_2)$ is continuous. Finally the map

$$(F, A) \in S(E) \times \mathrm{MALG} \to F|A \in S(E),$$

where $F|A = \{(x, y)\colon (x, y \in A \;\&\; xEy) \vee x = y\}$, is continuous.

One the other hand, the operation $(F_1, F_2) \mapsto F_1 \vee F_2$ from $S(E) \times S(E)$ to $S(E)$ is not continuous, if E is aperiodic. (Here $F_1 \vee F_2$ is the smallest equivalence relation containing both F_1, F_2.) To see this, first find $S \in [E]$ which is aperiodic, see [K4, 8.16]. Let $F_n = E_{S^{2^n}}$, so that the F_n are decreasing and $\bigcap_n F_n = id$, where *id* is the equality relation on X, thus $F_n \to id$. Let also $F = F_{S^3}$. Since for each n, 2^n and 3 are relatively prime, it is clear that $F_n \vee F = E_S$. On the other hand $id \vee F = E_{S^3} \neq E_S$.

Proposition 4.29. *The operation $(F_1, F_2) \mapsto F_1 \vee F_2$ from $S(E) \times S(E)$ to $S(E)$ is of Baire class 1.*

Proof. For each $T_1, T_2 \in [E]$ and $F \in S(E)$, let

$$A_{T_1,T_2,F} = \{x \colon (T_1(x), T_2(x)) \in F\}$$

(so that $A_{T,F} = A_{id,T,F}$). Since $A_{T_1,T_2,F} = T_1^{-1}(A_{T_2 T_1^{-1},F})$, it is clear that $F \mapsto A_{T_1,T_2,F}$ is continuous for every $T_1, T_2 \in [E]$.

In order to prove the proposition, it is enough to show that for any $T \in [E], \alpha < \beta$ in $\mathbb{R}$,

$$\{(F_1, F_2) \colon \alpha < \mu(A_{T,F_1 \vee F_2}) < \beta\}$$

is F_σ. Let $(T_i)_{i \in \mathbb{N}}$ be a generating sequence for E.

Let

$$\begin{aligned} x \in A^n_{T,F_1,F_2} \iff & \exists m \leq n \exists i_1, \ldots, i_{2m+1} \leq n (x \in A_{T_{i_1},F_1} \,\& \\ & x \in A_{T_{i_1},T_{i_2},F_2} \,\& \ \ldots \& \, x \in A_{T_{i_{2m}},T_{i_{2m+1}},F_1} \,\& \\ & x \in A_{T_{i_{2m+1}},T,F_2}). \end{aligned}$$

Then $A^n_{T,F_1,F_2} \subseteq A^{n+1}_{T,F_1,F_2}$ and $A_{T,F_1 \vee F_2} = \bigcup_n A^n_{T,F_1,F_2}$. So

$$\mu(A_{T,F_1 \vee F_2}) > \gamma \iff \exists n (\mu(A^n_{T,F_1,F_2}) > \gamma).$$

Since A^n_{T,F_1,F_2} is equal to

$$\bigcup_{m \leq n} \bigcup_{i_1,\ldots,i_{2m+1} \leq n} (A_{T_{i_1},F_1} \cap A_{T_{i_1},T_{i_2},F_2} \cap \cdots \cap A_{T_{i_{2m}},T_{i_{2m+1}},F_1} \cap A_{T_{i_{2m+1}},T,F_2}),$$

the map $(F_1, F_2) \mapsto A^n_{T,F_1,F_2}$ is continuous and thus the set $\{(F_1, F_2) \colon \gamma < \mu(A_{T,F_1 \vee F_2})\}$ is open, for every $\gamma \in \mathbb{R}$. It follows that

$$\{(F_1, F_2) \colon \mu(A_{T,F_1 \vee F_2}) < \beta\} = S(E)^2 \setminus \{(F_1, F_2) \colon \forall n (\beta - \tfrac{1}{n} < \mu(A_{T,F_1 \vee F_2}))\}$$

is F_σ and so $\{(F_1, F_2) \colon \alpha < \mu(A_{T,F_1 \vee F_2}) < \beta\}$ is F_σ. $\square$

4.6 The uniform topology

We will next discuss a stronger topology for $S(E)$. Recall that the topology on $S(E)$ is the smallest topology making the functions $F \mapsto A_{T,F}, T \in [E]$, from $S(E)$ to MALG, continuous. It is also the smallest topology making the functions $F \mapsto \mu(A_{T,F}), T \in [E]$, from $S(E)$ to $[0,1]$, continuous. This topology is induced by the equivalent metrics:

$$\tau(F_1, F_2) = \sum_{i=0}^{\infty} 2^{-(i+1)} \mu(A_{T_i,F_1} \Delta A_{T_i,F_2}),$$

$$\tau'(F_1, F_2) = \sum_{i=0}^{\infty} 2^{-(i+1)} \left| \mu(A_{T_i,F_1}) - \mu(A_{T_i,F_2}) \right|,$$

where $(T_i)_{i \in \mathbb{N}}$ is a dense sequence in $[E]$. Consider now the following two metrics:

$$\tau_\infty(F_1, F_2) = \sup_i \mu(A_{T_i,F_1} \Delta A_{T_i,F_2}) = \sup_{T \in [E]} \mu(A_{T,F_1} \Delta A_{T,F_2}),$$

$$\tau'_\infty(F_1, F_2) = \sup_i \left| \mu(A_{T_i,F_1}) - \mu(A_{T_i,F_2}) \right| = \sup_{T \in [E]} \left| \mu(A_{T,F_1}) - \mu(A_{T,F_2}) \right|.$$

Proposition 4.30. $\tau'_\infty \leq \tau_\infty \leq 3\tau'_\infty$.

Proof. Clearly $\tau'_\infty \leq \tau_\infty$. Let now $\tau'_\infty(F_1, F_2) = a$. We will show that $\tau_\infty(F_1, F_2) \leq 3a$. We have that $|\mu(A_{T,F_1}) - \mu(A_{T,F_2})| \leq a$, for all $T \in [E]$, so in particular for $S \in [F_1]$, $1 - \mu(A_{S,F_2}) \leq a$, i.e., $d(S, F_2) \leq a$. Now given $T \in [E]$, there is $S \in [F_1]$ such that $x \in A_{T,F_1} \implies S(x) = T(x)$ (see Proposition 4.2). Then by the last two paragraphs of the proof of Theorem 4.15,

$$\mu(A_{T,F_1} \setminus A_{T,F_2}) \leq d(S, F_2) \leq a$$

and also

$$\mu(A_{T,F_1} \setminus A_{T,F_2}) - \mu(A_{T,F_2} \setminus A_{T,F_1}) = \mu(A_{T,F_1}) - \mu(A_{T,F_2}),$$

therefore

$$\mu(A_{T,F_2} \setminus A_{T,F_1}) \leq \mu(A_{T,F_1} \setminus A_{T,F_2}) + a \leq 2a,$$

so $\mu(A_{T,F_1} \Delta A_{T,F_2}) \leq 3a$, thus $\tau_\infty(F_1, F_2) \leq 3a$. □

Thus $\tau_\infty, \tau'_\infty$ induce the same topology, which we call the **uniform topology** of $S(E)$. It clearly contains the topology of $S(E)$. It is easy to see that the metric τ_∞ (or equivalently τ'_∞) is complete. Indeed let (F_n) be τ_∞-Cauchy. Then it is also τ-Cauchy (where we can assume that τ is defined using a countable dense subgroup of $[E]$), so, by the proof of Theorem 4.13, there is $F \in S(E)$, such that $F_n \to F$ (in the topology of $S(E)$). Fix now $\epsilon > 0$ and let N be big enough, so that for $m, n \geq N$, we have $\tau'_\infty(F_m, F_n) \leq \epsilon$. Let $T \in [E]$ and then choose $N_0 > N$ such that for $m \geq N_0, |\mu(A_{T,F_m}) - \mu(A_{T,F})| \leq \epsilon$. Then for $n \geq N$ we have $|\mu(A_{T,F_n}) - \mu(A_{T,F})| \leq \left|\mu(A_{T,F_n}) - \mu(A_{T,F_{N_0}})\right| + \left|\mu(A_{T,F_{N_0}}) - \mu(A_{T,F})\right| \leq 2\epsilon$, thus $\tau'_\infty(F_n, F) \leq 2\epsilon$.

However the uniform topology is not, in general, separable.

Proposition 4.31. *Let E be aperiodic. Then the uniform topology on $S(E)$ is not separable.*

Proof. Let $F \subseteq E$ be aperiodic hyperfinite. Then there is free Borel action a of $\mathbb{Z}_2^{<\mathbb{N}}$ such that $E_a = F$ (see [K4, 8.10]). For Γ a subgroup of $\mathbb{Z}_2^{<\mathbb{N}}$ consider the subequivalence relation E_Γ induced by the restriction of the action a to Γ. There are clearly uncountably many such Γ and the map $\Gamma \mapsto E_\Gamma$ is injective. Suppose now that Γ is not contained in Δ and choose $\gamma \in \Gamma \setminus \Delta$. Then $\mu(A_{\gamma^a, E_\Gamma}) = 1$. On the other hand, by the freeness of a, there is no x such that $\gamma^a(x) = \delta^a(x)$, for some $\delta \in \Delta$. Thus $\mu(A_{\gamma^a, E_\Delta}) = 0$. It follows that $\tau'_\infty(E_\Gamma, E_\Delta) = 1$, thus the uncountable set consisting of the E_Γ's is discrete. □

Remark 4.32. Recall the metric σ on $S(E)$ defined in Remark 4.17. Then the topology induced by σ contains the uniform topology. In particular, the uniform topology is separable when restricted to the finite subequivalence relations of E.

5. Limits of sequences

The following shows how the limit of a convergent sequence in $S(E)$ is related to the members of the sequence.

Theorem 5.1. *Let $F_n, F \in S(E)$ and $F_n \to F$. Then for each i, there is an increasing sequence $n_0^{(i)} < n_1^{(i)} < \dots$, so that $(n_m^{(i+1)})_{m\in\mathbb{N}}$ is a subsequence of $(n_m^{(i)})_{m\in\mathbb{N}}$ and*

$$F = \bigcup_m \bigcap_{k\geq m} F_{n_k^{(m)}}.$$

Proof. Let $\{T_i\}_{i\in\mathbb{N}}$ be a countable subset of $[F]$, with $T_0 = id$, such that $xFy \iff \exists i(T_i(x) = y)$. We will define for each i an increasing sequence

$$n_0^{(i)} < n_1^{(i)} < \dots,$$

so that $(n_m^{(i+1)})_{m\in\mathbb{N}}$ is a subsequence of $(n_m^{(i)})_{m\in\mathbb{N}}$ and moreover if we put

$$R_{i,m} = \bigcap_{k\geq m} F_{n_k^{(i)}},$$

then for (almost) all x,

$$(x, T_i(x)) \in \bigcup_m R_{i,m}.$$

We construct $(n_m^{(i)})$, recursively on i.

To start with, take $n_m^{(0)} = m$. Assume now $(n_m^{(i)})$ is defined. We will next define $(n_m^{(i+1)})$.

Consider $T_{i+1} \in [F]$. Since $d(T_{i+1}, F) = 0$,

$$d(T_{i+1}, F_n) = \mu(\{x \colon (x, T_{i+1}(x)) \notin F_n\}) \to 0,$$

so we can find a subsequence $(n_m^{(i+1)})$ of $(n_m^{(i)})$ with $\mu(A_m) < 2^{-(m+1)}$, where

$$A_m = \{x\colon (x, T_{i+1}(x)) \notin F_{n_m^{(i+1)}}\}.$$

Thus, by the Borel-Cantelli Lemma, $\mu(\overline{\lim}_m A_m) = 0$, where $\overline{\lim}_m A_m = \bigcap_m \bigcup_{k\geq m} A_k$. Therefore, $\mu(\underline{\lim}_m(\sim A_m)) = 1$, where

$$\underline{\lim}_m B_m = \bigcup_m \bigcap_{k\geq m} B_k,$$

i.e., for almost all x, there is m such that for all $k \geq m$, $(x, T_{i+1}(x)) \in F_{n_k^{(i+1)}}$, thus $(x, T_{i+1}(x)) \in \bigcup_m R_{i+1,m}$.

Now note that $R_{i,m} \subseteq R_{i+1,m}$ and $R_{i,m} \subseteq R_{i,m+1}$, thus $R_{i,m} \subseteq R_{j,n}$ if $i \leq j, m \leq n$. So let $R_m = R_{m,m}$. Then clearly $\bigcup_{i,m} R_{i,m} = \bigcup_m R_m$ and $R_0 \subseteq R_1 \subseteq \dots$. Finally if $(x,y) \in F$, then for some $i, y = T_i(x)$, so $(x,y) \in \bigcup_m R_{i,m} \subseteq \bigcup_m R_m$, i.e., $F \subseteq \bigcup_m R_m$. We thus have $F \subseteq \bigcup_m \bigcap_{k\geq m} F_{n_k^{(m)}}$.

We will now verify that conversely

$$R = \bigcup_m \bigcap_{k\geq m} F_{n_k^{(m)}} \subseteq F.$$

Let $T \in [R]$ in order to show that $T \in [F]$. We have $\forall x \exists m[(x, T(x)) \in \bigcap_{k\geq m} F_{n_k^{(m)}}]$. Let

$$A_m = \left\{x\colon (x, T(x)) \in \bigcap_{k\geq m} F_{n_k^{(m)}}\right\},$$

so that $\bigcup_m A_m = X$. Now $F_{n_k^{(m)}} \to F$ as $k \to \infty$, so $A_{T,F_{n_k^{(m)}}} \xrightarrow{\text{MALG}} A_{T,F}$. Since $A_m \subseteq A_{T,F_{n_k^{(m)}}}$ for all $k \geq m$, by taking limits as $k \to \infty$, we obtain $A_m \subseteq A_{T,F}$, i.e., $x \in A_m \Rightarrow (x, T(x)) \in F$ and so $(x, T(x)) \in F$ for all x, i.e., $T \in [F]$. $\square$

Although the preceding result is sufficient for the subsequent applications, Le Maître in [LeM1], showed, using the description of the topology in Section 4.4, (1), that one has the following stronger form of Theorem 5.1 (similar to Theorem 5.6):

Let $F_n, F \in S(E)$ and $F_n \to F$. Then there is an increasing sequence $n_0 < n_1 < \ldots$, so that

$$F = \bigcup_m \bigcap_{k \geq m} F_{n_k}.$$

For $\mathcal{R}$ a class of measure preserving countable Borel equivalence relations on (X, μ), let

$$\mathcal{R}_{\downarrow} = \left\{ \bigcap_n F_n \colon F_0 \supseteq F_1 \supseteq \ldots, F_i \in \mathcal{R} \right\},$$

and

$$\mathcal{R}_{\uparrow} = \left\{ \bigcup_n F_n \colon F_0 \subseteq F_1 \subseteq \ldots, F_i \in \mathcal{R} \right\}.$$

Theorem 5.2. *Let $\mathcal{R} \subseteq S(E)$ be closed under finite intersections. Then*

$$\overline{\mathcal{R}} = (\mathcal{R}_{\downarrow})_{\uparrow}.$$

(where $\overline{\mathcal{R}}$ is the closure of $\mathcal{R}$ in $S(E)$).

Proof. Clearly $(\mathcal{R}_{\downarrow})_{\uparrow} \subseteq \overline{\mathcal{R}}$. The converse follows from Theorem 5.1, noting that $\bigcap_{k \geq m} F_{n_k^{(m)}}$ can be written as a decreasing intersection of relations in $\mathcal{R}$. □

Put

$$\mathcal{R}^* = (\mathcal{R}_{\downarrow})_{\uparrow}.$$

The preceding shows that if $\mathcal{R} \subseteq S(E)$ is closed under finite intersections, then $\mathcal{R}^* = \overline{\mathcal{R}}$ and thus $(\mathcal{R}^*)^* = \mathcal{R}^*$. Also note that if $\mathcal{R}$ is **hereditary**, i.e, closed under subequivalence relations, then $\overline{\mathcal{R}} = \mathcal{R}^* = \mathcal{R}_{\uparrow}$.

For any class $\mathcal{R}$ of measure preserving countable Borel equivalence relations on (X, μ) closed under finite intersections (not necessarily contained in some $S(E)$), we have that if $F \in (\mathcal{R}^*)^*$, then for some large enough E, $F \in (\mathcal{R}^*_E)^*$ with $\mathcal{R}_E = \mathcal{R} \cap S(E)$, so $F \in \mathcal{R}^*_E$. Thus $(\mathcal{R}^*)^* = \mathcal{R}^*$.

This has the following implication about arbitrary hereditary classes of equivalence relations (not necessarily within a fixed $S(E)$). It was originally proved (in a somewhat stronger form not requiring invariance of the measure) in Boykin-Jackson [BJ, page 116].

Corollary 5.3 (Boykin-Jackson [BJ]). *Let $\mathcal{R}$ be a hereditary class of measure preserving countable Borel equivalence relations on (X, μ). Then $\mathcal{R}_\uparrow$ is closed under taking unions of increasing sequences of relations, i.e., $(\mathcal{R}_\uparrow)_\uparrow = \mathcal{R}_\uparrow$.*

Proof. Let $S_n \in \mathcal{R}_\uparrow, S_0 \subseteq S_1 \subseteq \dots, E = \bigcup_n S_n$. Then if $\mathcal{R}_E = S(E) \cap \mathcal{R}$, we have that $(\mathcal{R}_E)_\uparrow = S(E) \cap \mathcal{R}_\uparrow$ and $S_n \in (\mathcal{R}_E)_\uparrow$, so $E \in \overline{\mathcal{R}_E}$, by Proposition 4.6, and thus $E \in \mathcal{R}_\uparrow$. □

Put also

$$\mathcal{R}_* = (\mathcal{R}_\uparrow)_\downarrow.$$

If $\mathcal{R}$ is closed under finite intersections, then $\mathcal{R}_* \subseteq \overline{\mathcal{R}} = \mathcal{R}^*$.

Problem 5.4. *If $\mathcal{R}$ is closed under finite intersections, is it true that $\mathcal{R}_* = \mathcal{R}^*$?*

We also have the following corollary of Theorem 5.1. Recall that id is the equality equivalence relation.

Corollary 5.5. *If $F_n \in S(E), F_n \to id$, then $\bigcap_n F_n = id$.*

Proof. By Theorem 5.1, there is an increasing sequence (n_i) with $\bigcap_i F_{n_i} = id$ and thus $\bigcap_n F_n = id$. □

We finally note the following for the uniform topology.

Theorem 5.6. *Let $F_n, F \in S(E)$ and $F_n \to F$ in the uniform topology. Then there is an increasing sequence $n_0 < n_1 < \dots$, so that*

$$F = \bigcup_m \bigcap_{k \geq m} F_{n_k}.$$

Proof. We have

$$\sup_{T \in [F]} \mu(X \setminus A_{T,F_n}) \to 0,$$

therefore let $n_0 < n_1 < \dots$ be such that for every $T \in [F], \mu(X \setminus A_{T,F_{n_m}}) < 2^{-(m+1)}$. Then for every $T \in [F], \mu(\underline{\lim}_m (A_{T,F_{n_m}})) = 1$, so for every $T \in [F]$ and (almost) all x there is m such that for all $k \geq m$, $x \in A_{T,F_{n_k}}$). It follows that

$$F \subseteq \bigcup_m \bigcap_{k \geq m} F_{n_k}.$$

The reverse inclusion follows as in the last part of the proof of Theorem 5.1.
□

6. The space of equivalence relations

We discuss here a topology on the space of *all* measure preserving countable Borel equivalence relations.

6.1 Coherence of topologies

We consider now the relation of the topologies of $S(E), S(F)$, when $E \subseteq F$.

Theorem 6.1. *Let $E \subseteq F$. Then $S(E)$ is a closed subset of $S(F)$ and the topology of $S(E)$ is the relative topology from $S(F)$.*

Proof. From Theorem 5.1 it is clear that $S(E)$ is a closed subset of $S(F)$.

Let τ be the relative topology of $S(E)$ and let σ be the topology of $S(E)$. We will use the description of the topology of $S(E)$ from Section 4.4, (1). Let M_E, M_F be the corresponding measures and let ν_F be a probability Borel measure equivalent to M_F. Then $\nu_F(E) > 0$, so let ν_E be the normalized measure on E given by $\nu_E(W) = \frac{\nu_F(W)}{\nu_F(E)}$, for Borel $W \subseteq E$. Then, since M_E is simply the restriction of M_F to E, clearly ν_E is equivalent to M_E. It follows that the identity map is a homeomorphism of $(S(E), \tau)$ with $(S(E), \sigma)$, so $\tau = \sigma$. $\square$

Denote by $\mathcal{E}$ the set of all measure preserving countable Borel equivalence relations on (X, μ) (where again we identify two equivalence relations if they agree a.e.). Thus $\mathcal{E} = \bigcup_{E \in \mathcal{E}} S(E)$. By the preceding Theorem 6.1, the topologies on $S(E), S(F)$ agree on $S(E) \cap S(F) = S(E \cap F)$ and $S(E \cap F)$ is closed on $S(E)$ and $S(F)$. So we can define the **weak topology** on $\mathcal{E}$ induced by the spaces $S(E)$, which is the topology on $\mathcal{E}$ defined by declaring that $U \subseteq \mathcal{E}$ is open iff $U \cap S(E)$ is open in $S(E)$ for all $E \in \mathcal{E}$. In particular $f\colon \mathcal{E} \to Y$, Y a topological space, is continuous if $f|S(E)\colon S(E) \to Y$ is continuous for all $E \in \mathcal{E}$. Also on each $S(E)$ the relative topology from $\mathcal{E}$ coincides with its topology and $S(E)$ is closed in $\mathcal{E}$. (For the general concept of weak topology on a set induced by topologies on families of subsets, see, e.g., [D, VI.8].)

We should also note here that for $E \subseteq F$, $S(E)$ is a retract of $S(F)$, with the retraction given by the map $R \in S(F) \mapsto R \cap E \in S(E)$. From this it follows that the map $E \in S(F) \mapsto S(E) \in \mathcal{F}^*(S(F))$, where $\mathcal{F}^*(S(F))$ is equipped with the Effros Borel structure, is a Borel map. To see this fix a countable dense subset $\{F_n\colon n \in \mathbb{N}\}$ of $S(F)$. Then $\{E \cap F_n\colon n \in \mathbb{N}\}$ is dense in $S(E)$ and the map $\Phi\colon S(F) \to S(F)^{\mathbb{N}}$ given by $\Phi(E)_n = E \cap F_n$ is Borel and gives for each $E \in S(F)$ a dense sequence in $S(E)$, which implies that the map $E \in S(F) \mapsto S(E) \in \mathcal{F}^*(S(F))$ is Borel. On the other hand, we do not know if the map $E \in S(F) \mapsto \{G \in S(F)\colon E \subseteq G\} \in \mathcal{F}^*(S(F))$ is Borel.

Finally we point out that each space $S(E)$ is contractible (to the equality relation) by the map $\varphi\colon S(E) \times [0,1] \to S(E)$ given by $\varphi(F,t) = F|[1, 1-t] \cup \{(x,x)\colon x \in (1-t, 1]\}$, where without loss of generality we assume that $X = [0,1]$ and μ is Lebesgue measure.

Remark 6.2. The question of the Borelness of the map $E \in S(F) \mapsto S(E) \in \mathcal{F}^*(S(F))$ is a special case of the following more general question: Let X be a Polish space and $\leq$ a closed (as a subset of X^2) partial ordering on X. Is the map $x \in X \mapsto I_x = \{y \in X\colon y \leq x\} \in \mathcal{F}^*(X)$ Borel (where again $\mathcal{F}^*(X)$ is equipped with the Effros Borel structure)?

The answer is in general negative. To see this, fix a Polish space Y and a closed subset $F \subseteq Y^2$ such that $\forall y \in Y \exists z \in Y (y,z) \in F$ but there is no Borel function $f\colon Y \to Y$ such that $\forall y \in Y (y, f(y)) \in F$ (see, e.g., [K, Exercise 18.17]).

Let $Y^\infty = \bigoplus_{n\geq 1} Y^n$ be the direct sum of the Y^n. Thus each Y^n is clopen in Y^∞. Let $X \subseteq Y^\infty$ be the closed subset of Y^∞ consisting of $Y^1 = \{(y)\colon y \in Y\}$, and for each $n \geq 2$ of all $(y_1, \dots, y_n)$ such that $(y_{n-1}, y_n) \in F, (y_{n-2}, y_{n-1}) \in F, \dots, (y_1, y_2) \in F$.

Finally define on X the partial ordering

$$s = (s_1, \dots, s_n) \leq t = (t_1, \dots, t_m) \iff n \geq m \mathbin{\&} s \supseteq t.$$

This is closed in X. Suppose now, towards a contradiction, that the map $x \mapsto I_x$ is Borel. Define $K\colon Y \to \mathcal{F}^*(Y^2)$ by $K(y) = I_{(y)} \cap (X \cap Y^2)$. Since

$X \cap Y^2$ is clopen in X, this is also a Borel map. Now

$$K(y) = \{(u,v) \in Y^2 \colon (u,v) \in F \;\&\; u = y\} = \{y\} \times F_y.$$

Since K is Borel, there is a Borel map $\phi\colon Y \to Y$ such that $\phi(y) \in K(y)$ and so there is a Borel map $f\colon Y \to Y$ such that $f(y) \in F_y$, contradicting our assumption about F.

On the other hand, it is easy to see that if $\leq$ is a closed *pre-linear* ordering on a Polish space X, then the map $x \mapsto I_x$ is Borel. Indeed for each open set $U \subseteq X$, let $P_U = \{x \in X \colon I_x \cap U \neq \emptyset\}$. We will check that P_U is Borel, in fact either open or closed. For that notice that for each $x \in X$, the set $J_x = \{y \in X \colon x < y\}$ is open. Then if U has a least element u_0, $P_U = \{x \colon u_0 \leq x\}$ is closed, while if U has no least element, then $P_U = \bigcup_{x \in U} J_x$ is open.

6.2 Properties of the weak topology

We will next give another description of the weak topology of $\mathcal{E}$. Consider the compact space $[0,1]^{\mathrm{Aut}(X,\mu)}$ with the product topology. Define

$$\Pi\colon \mathcal{E} \to [0,1]^{\mathrm{Aut}(X,\mu)}$$

by $\Pi(F)(T) = d(T,F)$. Since $[F] = \{T \colon d(T,F) = 0\}$, clearly Π is injective.

Proposition 6.3. *The map Π defined above is a homeomorphism of $\mathcal{E}$ with a subspace of $[0,1]^{\mathrm{Aut}(X,\mu)}$.*

Proof. Below denote by τ the weak topology of $\mathcal{E}$. We first verify that Π is continuous. Let $V = \bigcap_{i=1}^n V_i$ be a basic open set in $[0,1]^{\mathrm{Aut}(X,\mu)}$, where $V_i = \{p \in [0,1]^{\mathrm{Aut}(X,\mu)} \colon p(T_i) \in U_i\}$, with U_i open in $[0,1]$ and $T_1, T_2, \cdots, T_n \in \mathrm{Aut}(X,\mu)$. Then

$$\Pi^{-1}(V) = \{F \in \mathcal{E} \colon d(T_i, F) \in U_i, 1 \leq i \leq n\}.$$

Let $F \in \mathcal{E}$ be such that $T_i \in [F]$ for all $1 \leq i \leq n$. Then $\Pi^{-1}(V) \cap S(F)$ is open in $S(F)$. Since for each $E \in \mathcal{E}$ there is such an F containing E, it follows from Theorem 6.1 that $\Pi^{-1}(V) \cap S(E)$ is open in $S(E)$ for each $E \in \mathcal{E}$, so $\Pi^{-1}(V)$ is τ-open.

Conversely, we show that Π sends τ-closed sets to closed subsets of $\Pi(\mathcal{E})$ (in its relative topology from $[0,1]^{\mathrm{Aut}(X,\mu)}$), so Π^{-1} is also continuous.

Fix $\mathcal{F} \subseteq \mathcal{E}$ which is τ-closed. Let $(F_i)_{i\in I}$ be a net in $\mathcal{F}$ and $F \in \mathcal{E}$ be such that $\Pi(F_i) \to \Pi(F)$, i.e., $d(T, F_i) \to d(T, F), \forall T \in \mathrm{Aut}(X, \mu)$. We will show that $F \in \mathcal{F}$.

We inductively define an increasing sequence $E_0 \subseteq E_1 \subseteq \dots$ of elements of $\mathcal{E}$ and for each $n \in \mathbb{N}$ a countable dense subset $\{T_k^n\}_{k\in\mathbb{N}}$ of $[E_n]$ such that $\{T_k^n\}_{k\in\mathbb{N}} \subseteq \{T_k^{n+1}\}_{k\in\mathbb{N}}$, as follows:

(i) $E_0 = F$, $\{T_k^0\}_{k\in\mathbb{N}}$ is some dense subset of $[E_0]$,

(ii) Given E_n, $\{T_k^n\}_{k\in\mathbb{N}}$, for each $l \geq 1$, finite sequence $\overline{m} = (m_1, \dots, m_l) \in \mathbb{N}^l$, and $\epsilon \in \mathbb{Q}^+$, find $F^n_{\overline{m},\epsilon} \in \mathcal{F}$ such that

$$\left| d(T^n_{m_i}, F) - d(T^n_{m_i}, F^n_{\overline{m},\epsilon}) \right| < \epsilon, 1 \leq i \leq l.$$

Put

$$E_{n+1} = E_n \vee (\bigvee_{\overline{m},\epsilon} F^n_{\overline{m},\epsilon}),$$

where for a sequence of equivalence relations (F_j), $\bigvee_j F_j$ is the smallest equivalence relation containing all F_j. Finally let $\{T_k^{n+1}\}_{k\in\mathbb{N}}$ be a dense subset of $[E_{n+1}]$ containing $\{T_k^n\}_{k\in\mathbb{N}}$.

Let $E = \bigcup_n E_n$. Since $\mathcal{F} \cap S(E)$ is closed in $S(E)$, it is enough to show that F is in the closure of $\mathcal{F} \cap S(E)$ in $S(E)$. Since $\{T_k^n\}_{n,k}$ is dense in $[E]$, a basic open nbhd of F in $S(E)$ is of the form

$$U = \bigcap_{i=1}^{l} \{F' \in S(E) \colon\ |d(S_i, F') - d(S_i, F)| < \epsilon\},$$

for some $S_1, \dots, S_l \in \{T_k^n\}_{n,k}$ and $\epsilon \in \mathbb{Q}^+$. Then for large enough n, we have that $S_1, \dots, S_l \in \{T_k^n\}_k$, say $S_i = T^n_{m_i}, 1 \leq i \leq l$. Put $\overline{m} = (m_1, \dots, m_l)$. Then by construction $F^n_{\overline{m},\epsilon} \in \mathcal{F} \cap U$ and the proof is complete. □

Thus $\mathcal{E}$ can be viewed as a subspace of $[0, 1]^{\mathrm{Aut}(X,\mu)}$, so, in particular, it is Hausdorff. On the other hand it is neither separable or first countable.

Proposition 6.4. *The weak topology on $\mathcal{E}$ is not separable.*

Proof. The closure of any countable set $\{F_n\} \subseteq \mathcal{E}$ is clearly contained in $S(\bigvee_n F_n)$. □

Proposition 6.5. *The weak topology of $\mathcal{E}$ is not first countable.*

Proof. We will use the following lemma. Recall that for $T \in \mathrm{Aut}(X,\mu)$, E_T is the equivalence relation generated by T.

Lemma 6.6. *Let $F \in \mathcal{E}, S_1, S_2, \dots \in \mathrm{Aut}(X,\mu)$ and $T \in \mathrm{Aut}(X,\mu)$ be such that $E_T \perp (F \vee (\bigvee_{i=1}^{\infty} E_{S_i}))$. Then for every $i \geq 1$, $d(S_i, F \vee E_T) = d(S_i, F)$.*

Proof. By definition of $\perp$, $A_{S_i, F\vee E_T} = A_{S_i,F}$. □

Assume now, towards a contradiction, that $\mathcal{E}$ is first countable. Fix $F \in \mathcal{E}$ and let $\{U_n\}$ be a local basis at F. Then, for each n, there is a sequence $T_1^n, \dots, T_{k_n}^n \in \mathrm{Aut}(X,\mu)$ and open sets $V_i^n, \dots, V_{k_n}^n$ in $[0,1]$ such that

$$F \in \bigcap_{i=1}^{k_n} \{F' \colon d(T_i^n, F') \in V_i^n\} \subseteq U_n.$$

Let $R = F \vee \bigvee_{i \leq k_n, n\in\mathbb{N}} E_{T_i^n}$. The set $\{T \in \mathrm{Aut}(X,\mu) \colon E_T \perp R\}$ is comeager in the weak topology of $\mathrm{Aut}(X,\mu)$ (see Conley-Miller [CM, Theorem 8]), so fix aperiodic $T \in \mathrm{Aut}(X,\mu)$ with $E_T \perp R$. Then $d(T,F) = 1$. Put

$$U = \{F' \in \mathcal{E} \colon d(T,F') > \epsilon\},$$

where $0 < \epsilon < 1$. Then $F \in U$, so for some n, $F \in U_n \subseteq U$ and thus $F \in \bigcap_{i=1}^{k_n} \{F' \colon d(T_i^n, F') \in V_i^n\} \subseteq U$. Put $F' = F \vee E_T$. Then, since $E_T \perp R$, we have by Lemma 6.6 that $d(T_i^n, F') = d(T_i^n, F)$, therefore $d(T_i^n, F') \in V_i^n$ for $1 \leq i \leq k_n$. Thus $F' \in U_n \subseteq U$, so $d(T,F') > \epsilon$. But $T \in [F']$, so $d(T,F') = 0$, a contradiction. □

6.3 Parametrization by actions

To see another aspect of the global structure of $\mathcal{E}$, consider the Polish space $A(\mathbb{F}_\infty, X, \mu)$ with the weak topology. The map $a \mapsto E_a$ is a surjection from $A(\mathbb{F}_\infty, X, \mu)$ to $\mathcal{E}$ and provides a canonical parametrization of $\mathcal{E}$. Let

$$a \sim_{\mathbb{F}_\infty} b \iff E_a = E_b$$

be the associated equivalence relation, so that $\mathcal{E} = A(\mathbb{F}_\infty, X, \mu)/\sim_{\mathbb{F}_\infty}$.

Proposition 6.7. *$\sim_{\mathbb{F}_\infty}$ is $F_{\sigma\delta}$.*

Proof. We use below letters β, γ, δ for elements of $\mathbb{F}_\infty$ and a, b for elements of $A(\mathbb{F}_\infty, X, \mu)$. We will verify that the negation of $\sim_{\mathbb{F}_\infty}$ is $G_{\delta\sigma}$. For this it is enough to check that for each $\gamma, \epsilon > 0$ the relation

$$P(a, b) \iff \mu(\{x \colon \forall\delta(\gamma^a(x) \neq \delta^b(x))\}) \geq \epsilon$$

is G_δ and for this it suffices to check that for each fixed $\delta_1, \dots, \delta_n$ the relation

$$Q(a, b) \iff \mu(\{x \colon \forall 1 \leq i \leq n(\gamma^a(x) \neq \delta_i^b(x))\}) \geq \epsilon$$

is G_δ. Since the maps $a \mapsto \beta^a$ from $A(\mathbb{F}_\infty, X, \mu)$ to $\mathrm{Aut}(X, \mu)$ (with the weak topologies) are continuous, this reduces to showing that the relation $R \subseteq \mathrm{Aut}(X, \mu)^{n+1}$ given by

$$R(T, S_1, \dots, S_n) \iff \mu(\bigcap_{i=1}^{n} \mathrm{supp}(T^{-1}S_i)) \geq \epsilon$$

is G_δ, where as usual

$$\mathrm{supp}(T) = \{x \colon T(x) \neq x\}.$$

This is clear, since the map

$$(U_1, \dots, U_n) \in \mathrm{Aut}(X, \mu) \mapsto (\mathrm{supp}(U_1), \dots, \mathrm{supp}(U_n)) \in \mathrm{MALG}^n$$

is of Baire class 1 (see [K, page 4]). $\square$

Below let E_{ctble} be the equivalence relation on $P^{\mathbb{N}}$, where P is an uncountable Polish space, given by

$$(x_n) E_{\mathrm{ctble}} (y_n) \iff \{x_n \colon n \in \mathbb{N}\} = \{y_n \colon n \in \mathbb{N}\}.$$

It is well known that this is a non-smooth equivalence relation and moreover it is $F_{\sigma\delta}$-complete (as a set of pairs). Below for Borel equivalence relations E, F in Polish spaces X, Y, we let $E \leq_c F$ mean that there is a continuous reduction from E to F.

Theorem 6.8. *$E_{\mathrm{ctble}} \leq_c \sim_{\mathbb{F}_\infty}$, so, in particular, $\sim_{\mathbb{F}_\infty}$ is $F_{\sigma\delta}$-complete (as a set of pairs) and non-smooth.*

Proof. Let $R_n \subseteq \mathrm{Aut}(X,\mu)^n$ be defined by

$$R_n(T_1,\dots,T_n) \iff \forall 1 \le i \le n(E_{T_i} \perp E_{T_1,\dots,T_{i-1},T_{i+1},\dots,T_n}).$$

By Conley-Miller [CM, Theoem 8] and the Kuratowski-Ulam Theorem, a simple induction on n shows that each R_n is comeager in $\mathrm{Aut}(X,\mu)^n$. Thus by the Kuratowski-Mycielski Theorem (see [K2, 19.1]), there is a Cantor set $P \subseteq \mathrm{Aut}(X,\mu)$ so that for any distinct $T_1,\dots,T_n \in P$ we have $R_n(T_1,\dots,T_n)$.

Define now $f\colon P^{\mathbb{N}} \to A(\mathbb{F}_\infty, X, \mu)$ by $f((T_i)) = a$, where $\gamma_i^a = T_i$, with (γ_i) free generators of $\mathbb{F}_\infty$. Clearly f is continuous and a reduction of E_{ctble} to $\sim_{\mathbb{F}_\infty}$. □

It can be also shown that $\sim_{\mathbb{F}_\infty}$ is Borel reducible to an equivalence relation induced by a Borel action of a Polish group. In fact, by using a slightly different parametrization of $\mathcal{E}$, the associated equivalence relation is again $F_{\sigma\delta}$ and induced by a continuous action of a Polish group (see Törnquist [T, page 33]).

The preceding show that it is not possible to find a "definable" injection of $\mathcal{E}$ into a standard Borel space, so in particular $\mathcal{E}$ does not admit any "definable" separable metrizable topology. The following remains open:

Problem 6.9. *What is the complexity of the equivalence relation (as a set of pairs) on the space $A(\Gamma, X, \mu)$ (in the weak topology) given by*

$$a \sim_\Gamma b \iff E_a = E_b,$$

for other groups Γ, e.g., $\Gamma = \mathbb{Z}$?

Problem 6.10. *Determine the complexity of the equivalence relation $\sim_{\mathbb{F}_\infty}$ in the hierarchy of Borel equivalence relations under Borel reducibility.*

6.4 The inclusion poset

We finally note an interesting property of the poset $(\mathcal{E}, \subseteq)$. We start with the following simple observation.

Let $(P, \le)$ be an upper semilattice having the following two properties: (i) there is no strictly increasing ω_1 sequence in P and (ii) every increasing ω sequence in P has a least upper bound. Then for every function $f\colon P \to$

P and every $p \in P$, there is $q \geq p$ such that $f(q) \leq q$. Indeed if this fails, then for some p_0 and all $q \geq p_0$ we have that $q^* = q \vee f(q) > q$, which using (ii) above produces a strictly increasing ω_1 sequence. Moreover note that if f is **ω-continuous**, i.e., for any increasing sequence (p_n) we have that $f(\mathrm{lub}\ p_n) = \mathrm{lub} f(p_n)$, then the set $\{p \in P\colon f(p) \leq p\}$ is **ω-closed**, i.e., closed under suprema of increasing sequences, and cofinal.

In the following paragraph, we work in $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$.

By a result of Harrington [H] there is no injective ω_1 sequence of $F_{\sigma\delta}$ sets and thus condition (i) above holds for $(\mathcal{E}, \subseteq)$. Clearly condition (ii) is also true. Thus for every $f\colon \mathcal{E} \to \mathcal{E}$ and every $E \in \mathcal{E}$, there is $F \supseteq E$, with $f(F) \subseteq F$. An interesting example of such an f is defined as follows. Fix a measure preserving bijection φ of X^2 (with the product measure) with X and let $f(E)$ be the image of $E \times E$ by φ. Clearly f is ω-continuous. It follows that there is an ω-closed, cofinal set of E for which $E \times E$ is isomorphic to a subequivalence relation of E.

We work next in the stronger theory $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$.

Let for any uncountable Polish space Z, $\mathcal{P}_{\aleph_1}(Z)$ be the set of all countable subsets of Z. Solovay [S] has shown that the set $\mathcal{P}_{\aleph_1}(Z)$ admits a non-principal, countably complete ultrafilter $\mathcal{U}$ defined by:

$$A \in \mathcal{U} \iff A \text{ contains an } \omega\text{-closed, cofinal subset.}$$

(Here a subset $C \subseteq \mathcal{P}_{\aleph_1}(Z)$ is called ω-closed if for any $S_0 \subseteq S_1 \subseteq \dots$, with $S_n \in C, \forall n$, we have that $\bigcup_n S_n \in C$. It is cofinal if for any $S \in \mathcal{P}_{\aleph_1}(Z)$ there is $T \in C$ with $S \subseteq T$.)

We can use this to define a non-principal, countably complete ultrafilter on $\mathcal{E}$ as follows: For each $S \in \mathcal{P}_{\aleph_1}(\mathrm{Aut}(X, \mu))$, let E_S be the equivalence relation generated by S. Then for every $\mathcal{R} \subseteq \mathcal{E}$, put

$$\mathcal{R} \in \mathcal{U}_{\mathcal{E}} \iff \{S\colon E_S \in \mathcal{R}\} \in \mathcal{U}.$$

It is easy to see that if $\mathcal{R} \subseteq \mathcal{E}$ is ω-closed and cofinal in $(\mathcal{E}, \subseteq)$, then $\mathcal{R} \in \mathcal{U}_{\mathcal{E}}$, thus $\mathcal{U}_{\mathcal{E}}$ contains the countably complete filter of sets containing an ω-closed, cofinal subset of $\mathcal{E}$. The following is open:

Problem 6.11. *Is the filter generated by the ω-closed, cofinal subsets of $\mathcal{E}$ an ultrafilter? Equivalently, is $\mathcal{U}_{\mathcal{E}}$ equal to that filter?*

In any case, the ultrafilter $\mathcal{U}_{\mathcal{E}}$ provides a natural way to define a notion of “largeness" for sets of equivalence relations. For example, the class of ergodic equivalence relations is “large", i.e., belongs to $\mathcal{U}_{\mathcal{E}}$ (and we will see in Chapter 11 that so is the class of richly ergodic ones, being ω-closed, cofinal). On the other hand, the class of hyperfinite equivalence relations is “small", i.e., is not in $\mathcal{U}_{\mathcal{E}}$.

Remark 6.12. The above can be also viewed as results concerning “definable" functions and sets in $\mathcal{E}$, where we interpret “definable" as meaning “belonging to some inner model of $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$ or $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$ containing the set of reals $\mathbb{R}$" and working in a strong enough large cardinal extension of ZFC .

We conclude this section by pointing out the following unboundedness property of $\mathcal{E}$: There is no $E \in \mathcal{E}$ such that for every $F \in \mathcal{E}$ there is a subequivalence relation F' of E which is isomorphic to F. This follows from a result of Ozawa, see [K, page 29].

7. Relations with the space of actions

Let Γ be a countable group and let $(A(\Gamma, X, \mu), u)$ be the space of measure preserving actions of Γ on (X, μ) with the uniform topology u and consider its closed subspace $A(\Gamma, E)$. For each $a \in A(\Gamma, E)$, let $E_a \in S(E)$ be the equivalence relation induced by a. We also let $E_{T_1, T_2, \ldots}$ be the equivalence relation induced by $T_1, T_2, \ldots$ in $\mathrm{Aut}(X, \mu)$.

If $\Gamma = \mathbb{F}_\infty$, the map $a \mapsto E_a$ gives a parametrization of $S(E)$ by $A(\mathbb{F}_\infty, E)$, i.e., a surjective map from $A(\mathbb{F}_\infty, E)$ onto $S(E)$. By Theorem 4.28, and the paragraph preceding it, we have the following selection result.

Theorem 7.1. *There is a continuous map $\Psi\colon S(E) \to A(\mathbb{F}_\infty, E)$ such that for $F \in S(E)$, $E_{\Psi(F)} = F$.*

We now have:

Theorem 7.2. *The map $a \in A(\Gamma, E) \mapsto E_a \in S(E)$ is of Baire class 1.*

Proof. A subbasis for the weak topology of $S(E)$ consists of the sets of the form

$$\{F \in S(E) \colon d(T, F) \in (a, b)\},$$

where $T \in D$, with D a countable dense subset of $[E]$, and $a < b$ rationals. It is thus enough to show that

$$\{a \in A(\Gamma, E) \colon d(T, E_a) \in (a, b)\}$$

is F_σ and for that it suffices to show that for any such T and $r > 0$

$$\{a \in A(\Gamma, E) \colon d(T, E_a) \geq r\}$$

is closed.

So assume $a_n \to a$ in $A(\Gamma, E)$ and $d(T, E_{a_n}) \geq r$. Let $U \in E_a$ be such that $d(T,U) = d(T, E_a)$. Then $d(T, E_{a_n}) \leq d(T,U) + d(U, E_{a_n})$.

Claim. $d(U, E_{a_n}) \to 0$.

Granting this $r \leq \overline{\lim}_n d(T, E_{a_n}) \leq d(T,U) = d(T, E_a)$.

Proof of claim. Fix $\epsilon > 0$. Find next a Borel partition $\{A_n\}_{n=1}^{\infty}$ of X and elements $\{\gamma_n\}_{n=1}^{\infty}$ of Γ with $U = \bigsqcup_\gamma \gamma_n^a | A_n$. Let N be large enough so that $\sum_{n>N} \mu(A_n) < \epsilon$. Then let M be large enough, so that $d(\gamma_i^a, \gamma_i^{a_n}) < \frac{\epsilon}{N}$, if $i \leq N$ and $n > M$. Let for $i \leq N, n > M$

$$B_i^n = \{x \in A_i \colon \gamma_i^a(x) = \gamma_i^{a_n}(x)\}.$$

Then $B_i^n \subseteq A_i$ and $\mu(A_i \setminus B_i^n) < \frac{\epsilon}{N}$, so if

$$B = \bigcup_{i>N} A_i \cup \bigcup_{i \leq N} (A_i^n \setminus B_i^n),$$

then $\mu(B) < 2\epsilon$. If $x \notin B$, then $U(x) = \gamma_i^{a_n}(x)$ for some $i \leq N$, so $(x, U(x)) \in E_{a_n}$. Thus $\{x \colon (x, U(x)) \notin E_{a_n}\} \subseteq B$, so

$$d(U, E_{a_n}) = \mu(\{x \colon (x, U(x)) \notin E_{a_n}\}) < 2\epsilon$$

for all $n > M$ and we are done. □

Corollary 7.3. *Let $\Gamma = \mathbb{F}_\infty$. Let $\mathcal{P}$ be a property of equivalence relations such that*

$$\mathcal{P}_E^* = \{a \in A(\Gamma, E) \colon E_a \in \mathcal{P}\}$$

is Borel in $A(\Gamma, E)$. Then $\mathcal{P}_E = \mathcal{P} \cap S(E)$ is Borel in the topology of $S(E)$.

Proof. For $F \in S(E)$,

$$\begin{aligned} F \in \mathcal{P} &\iff \exists a \in A(\Gamma, E)\ (E_a = F \ \&\ a \in \mathcal{P}^*) \\ &\iff \forall a \in A(\Gamma, E)\ (E_a = F \Rightarrow a \in \mathcal{P}^*). \end{aligned}$$

Since $a \mapsto E_a$ is Borel this shows that $\mathcal{P} \cap S(E)$ is both analytic and co-analytic, thus Borel. □

In particular, taking again $\Gamma = \mathbb{F}_\infty$, suppose $\mathcal{P}$ is a property of equivalence relations such that $\{a \in A(\Gamma, X, \mu) \colon E_a \in \mathcal{P}\}$ is Borel in the topology of $A(\Gamma, X, \mu)$. Since this is contained in the uniform topology of $A(\Gamma, X, \mu)$, this set is Borel in the uniform topology of $A(\Gamma, X, \mu)$ and it follows that $\{a \in A(\Gamma, E) \colon E_a \in \mathcal{P}\}$ is Borel in (the uniform topology of) $A(\Gamma, E)$. Therefore $\mathcal{P}_E$ is Borel in $S(E)$.

Problem 7.4. *For which countable Γ is the map $a \in A(\Gamma, E) \mapsto E_a \in S(E)$ continuous, for each E?*

Anush Tserunyan and Robin Tucker-Drob found the first examples that showed that this map is not always continuous.

(1) (R. Tucker-Drob) Take $\Gamma = \mathbb{F}_\infty$ with free generating set $\{\gamma_m\}_{m\in\mathbb{N}}$. Fix any two transformations $S, T \in \mathrm{Aut}(X, \mu)$ such that $E_S \vee E_T \neq E_T$. Let $E = E_S \vee E_T$ and define $a_n \in A(\Gamma, E)$ by

$$\gamma_m^{a_n} = \begin{cases} S & \text{if } m > n, \\ T & \text{if } m \leq n. \end{cases}$$

Also define $a \in A(\Gamma, E)$ by $\gamma_m^a = T$ for all m. Clearly $E_{a_n} = E$ and $E_a = E_T$. Also a_n converges uniformly to a. On the other hand, the constant sequence $E_{a_n} = E$ does not converge to $E_a = E_T$.

(2) (A. Tserunyan) Take $\Gamma = \mathbb{F}_\infty$ with free generating set $\{\gamma_m\}_{m\in\mathbb{N}}$. Let a be the usual shift action of Γ on 2^Γ and E_a the induced equivalence relation. Let T be the measure preserving automorphism on 2^Γ such that $T(x)(\gamma) = 1 - x(\gamma)$, for all $\gamma \in \Gamma$. Let a_n be the action of Γ such that

$$\gamma_m^{a_n} = \begin{cases} \gamma_m^a & \text{if } m \leq n, \\ T & \text{if } m > n. \end{cases}$$

Let E be large enough so that all γ_n^a, for all $\gamma \in \Gamma$, and T are in $[E]$. Then $a_n \to a$ uniformly but E_{a_n} does not converge to E_a, since $d(T, E_{a_n}) = 0$ but $d(T, E_a) = 1$.

(3) (R. Tucker-Drob) Take $\Gamma = \mathbb{Z}$. Let (S, T) be any free action of $\mathbb{Z}^2$ on (X, μ) and fix some ergodic equivalence relation E whose full group contains the transformations S and T. Fix also sequences $\epsilon_n > 0$, $\epsilon_n \to 0$ and $k_n \in \mathbb{N}$, $k_n \to \infty$. By the Rokhlin Lemma for $\mathbb{Z}^2$ actions, for each n we can find a set $B_n = B \subseteq X$ such that the sets $S^iT^j(B)$, $0 \leq i, j < k_n$, are pairwise disjoint, and satisfy $\mu(C) > 1 - \epsilon_n$, where

$$C = \bigsqcup_{0 \leq i,j < k_n} S^iT^j(B).$$

Define the transformation S_n as follows: view powers of the transformation S as successively moving the block B upward and view powers of T as moving B horizontally to the right. We thus have a square structure consisting of k_n-many rows and k_n-many columns, and where B is in the bottom left corner. Define S_n to be equal to S on all rows except for the top. On the top row define

$$S_n(x) = \begin{cases} TS^{-(k_n-1)}(x) & \text{if } x \in S^{k_n-1}T^j(B) \text{ where } j < k_n - 1, \\ T^{-(k_n-1)}S^{-(k_n-1)}(x) & \text{if } x \in S^{k_n-1}T^{k_n-1}(B). \end{cases}$$

Thus, on C, S_n is a cyclic permutation of the blocks $\{S^iT^j(B)\}_{0\leq i,j<k_n}$ with $E_{S_n}|C \subseteq E_{(S,T)}|C \subseteq E|C$. We extend S_n to all of X so that it has period k_n^2 and is in $[E]$. Then it is clear that $S_n \to S$ uniformly. Since (S,T) defines a free action of $\mathbb{Z}^2$ we have $d(T, E_S) = 1 - \mu(\{x \in X : (x, T(x)) \in E_S\}) = 1$. On the other hand, if $x \in C$ is in any column except the last, then $S_n^{k_n}(x) = T(x)$, so that $(x, T(x)) \in E_{S_n}$ and thus $d(T, E_{S_n}) = 1 - \mu(\{x : (x, T(x)) \in E_{S_n}\}) \to 0$. This shows that E_{S_n} does not converge to E_S.

Assume now that S is ergodic. We can use the sequence $\{S_n\}$ to define a new sequence $\{Q_n\} \subseteq [E]$ of ergodic transformations which converge uniformly to S and also satisfy $\mu(\{x : T(x) = Q_n^{k_n}(x)\}) \to 1$, so that E_{Q_n} does not converge to E_S in $S(E)$. Let $\eta_n > 0$ be chosen so that $k_n\eta_n \to 0$. Since E is ergodic, any two transformations in $[E]$ of period k_n^2 are isomorphic via an element of $[E]$, so by the Uniform Approximation Theorem (see [K, 3.3]), for each n there exists an ergodic transformation $Q_n \in [E]$ such that $d_u(S_n, Q_n) \leq \frac{1}{k_n^2} + \eta_n$. Then $Q_n \to S$ uniformly (since S_n converges to S uniformly), and $d_u(Q_n^{k_n}, S_n^{k_n}) \leq k_n(\frac{1}{k_n^2} + \eta_n) = \frac{1}{k_n} + k_n\eta_n \to 0$, so $\mu(\{x : T(x) = Q_n^{k_n}(x)\}) \to 1$.

It turns out now that we have the following general fact.

Theorem 7.5. *Let E be ergodic. Let Γ be a countable infinite amenable group. Then the map $a \in A(\Gamma, E) \mapsto E_a \in S(E)$ is not continuous.*

Proof. By Dye and Ornstein-Weiss (see [KM, 10.7]), let $T \in [E]$ be mixing and let $a \in A(\Gamma, E)$ be such that $E_a = E_T$. Let $S = T^2$. Then S is ergodic and $E_S \subsetneq E_T$. Again by Ornstein-Weiss, there is a free ergodic $b \in A(\Gamma, E)$ such that $E_b = E_S$. By Foreman-Weiss [FW, proof of Claim 19], there is a sequence $S_0, S_1, \dots \in [E_a \vee E_b] = [E_a]$ such that $a_n = S_n a S_n^{-1} \to b$ uniformly. But $E_{a_n} = E_a \neq E_b$. $\square$

We now define a stronger topology than the uniform topology on the space $A(\Gamma, X, \mu)$ (see [K, Remark in page 103]). It is induced by the complete metric

$$\delta_{\Gamma,\infty}(a,b) = \sup_{\gamma\in\Gamma} d_u(\gamma^a,\gamma^b).$$

The main fact is that the map $a \in A(\Gamma, E) \mapsto E_a \in S(E)$ is Lipschitz in the metrics $\delta_{\Gamma,\infty}, \tau_\infty$ (defined in Section 4.6). Below recall that $[[E]]$ is the full pseudogroup of E, i.e., the set of all partial Borel bijections $\varphi\colon A \to B$ with $\varphi(x)Ex, \forall x \in A$. As usual we identify two such partial bijections if they agree μ-a.e.

Theorem 7.6. *For any countable group Γ and any $a, b \in A(\Gamma, E)$,*

$$\tau_\infty(E_a, E_b) \leq 80\delta_{\Gamma,\infty}(a,b).$$

In particular, $a \in A(\Gamma, E) \mapsto E_a \in S(E)$ is continuous from the $\delta_{\Gamma,\infty}$-topology on $A(\Gamma, E)$ to the uniform topology of $S(E)$ (and thus to the topology of $S(E)$).

Proof. We will show that for any $\delta > 0$,

$$\delta_{\Gamma,\infty}(a,b) < \frac{\delta^2}{2} \implies \tau_\infty(E_a, E_b) \leq 40\delta^2.$$

Assume $\delta_{\Gamma,\infty}(a,b) < \frac{\delta^2}{2}$ and fix $T \in [E]$. By [K, Remark in page 103], there is $\varphi\colon A \to B, \varphi \in [[E]]$ such that A is a-invariant, B is b-invariant, $\varphi(a|A)\varphi^{-1} = b|B$, $\mu(A) > 1 - 16\delta^2$ and $\mu(\{x \in A\colon \varphi(x) \neq x\}) \leq 4\delta^2$.

Put

$$A' = \{x \in A\colon \varphi(x) = x\} = \{x \in B\colon \varphi^{-1}(x) = x\},$$
$$A'' = \{x \in A'\colon T(x) \in A'\}.$$

Suppose now that $(x, T(x)) \in E_a$ and $x \in A''$. Then there is $\gamma \in \Gamma$ such that $\gamma^a(x) = T(x)$, so, as $x, T(x) \in A'$, we have

$$\gamma^b(x) = \gamma^b(\varphi(x)) = \varphi(\gamma^a(x)) = \varphi(T(x)) = T(x),$$

so $(x, T(x)) \in E_b$. Similarly let $(x, T(x)) \in E_b$ and $x \in A''$. Then there is $\gamma \in \Gamma$ such that $\gamma^b(x) = T(x)$, so, as $x, T(x) \in A'$, we have

$$T(x) = \varphi^{-1}(T(x)) = \varphi^{-1}(\gamma^b(x)) = \gamma^a(\varphi^{-1}(x)) = \gamma^a(x),$$

thus $(x, T(x)) \in E_a$.

It follows that $A_{T,E_a} \Delta A_{T,E_b} \subseteq X \setminus A''$, and, since $A'' = A' \cap T^{-1}(A')$, we have $\mu(X \setminus A'') \leq 2\mu(X \setminus A')$. Also $X \setminus A' \subseteq (X \setminus A) \cup \{x \in A\colon \varphi(x) \neq x\}$, so $\mu(X \setminus A') < 16\delta^2 + 4\delta^2 = 20\delta^2$. Thus $\mu(A_{T,E_a} \Delta A_{T,E_b}) < 40\delta^2$ and, since T was arbitrary, $\tau_\infty(E_a, E_b) \leq 40\delta^2$. $\square$

It is known that when Γ has property (T) the $\delta_{\Gamma,\infty}$-topology on $A(\Gamma, E)$ coincides with the (uniform) topology of $A(\Gamma, E)$ (see again [K, Remark in page 103]), so we have the following result originally proved by R. Tucker-Drob:

Corollary 7.7 (Tucker-Drob)**.** *If Γ has property* (T)*, then the map $a \in A(\Gamma, E) \mapsto E_a \in S(E)$ is continuous.*

In view of Theorem 7.5 and Corollary 7.7, one can consider the following more precise version of Problem 7.4.

Problem 7.8. *Let Γ be an infinite group. Is it true that the map $a \in A(\Gamma, E) \mapsto E_a \in S(E)$ is continuous for every E iff the group Γ has property* (T)*.*

Finally we have:

Proposition 7.9. *Assume $a_n \to a$ in $A(\Gamma, E)$ and for every $T \in [E]$,*

$$\mu(\{x\colon \neg T(x) E_a x \;\&\; T(x) E_{a_n} x\}) \to 0.$$

Then $E_{a_n} \to E_a$. In particular this holds if $E_{a_n} \subseteq E_a$.

Proof. We have to show that for $T \in [E], d(T, E_{a_n}) \to d(T, E_a)$.

Let now $U \in [E_a]$ be such that

$$T(x) E_a x \Rightarrow U(x) = T(x) \text{ (therefore } d(T, E_a) = d(U, T))$$

Let

$$\begin{aligned}
B_n &= \{x\colon \neg T(x) E_{a_n}(x) \;\&\; T(x) E_a x\} \\
C_n &= \{x\colon T(x) E_{a_n} x \;\&\; U(x) E_{a_n} x \;\&\; \neg T(x) E_a x\} \\
D_n &= \{x\colon T(x) E_{a_n} x \;\&\; \neg U(x) E_{a_n} x \;\&\; \neg T(x) E_a x\} \\
E_n &= \{x\colon \neg T(x) E_{a_n} x \;\&\; U(x) E_{a_n} x \;\&\; \neg T(x) E_a x\} \\
F_n &= \{x\colon \neg T(x) E_{a_n} x \;\&\; \neg U(x) E_{a_n} x \;\&\; \neg T(x) E_a x\}.
\end{aligned}$$

Let b_n, c_n, d_n, e_n, f_n be the measure of these sets, resp.

Then

$$\begin{aligned}
d(T, E_{a_n}) &= b_n + e_n + f_n \\
d(U, E_{a_n}) &= b_n + d_n + f_n
\end{aligned}$$

and

$$d(T, E_a) = c_n + d_n + e_n + f_n$$

Therefore $d(T, E_{a_n}) - d(T, E_a) = b_n - c_n - d_n$.

Now as $d(U, E_{a_n}) \to 0$ (by the Claim in Theorem 7.2), we have that $b_n + d_n \to 0$.

So it is enough to show that $c_n \to 0$. But

$$\begin{aligned} C_n &= \{x \colon T(x)E_{a_n}x \;\&\; U(x)E_{a_n}x \;\&\; \neg T(x)E_a x\} \\ &\subseteq \{x \colon T(x)E_{a_n}x \;\&\; \neg T(x)E_a x\}, \end{aligned}$$

so $c_n \to 0$ by hypothesis. $\square$

8. Complexity calculations

We now discuss the complexity of various classes of equivalence relations. For any class $\mathcal{R}$ of measure preserving countable Borel equivalence relations and any given such relation E, we denote by

$$\mathcal{R}_E = \mathcal{R} \cap S(E),$$

the set of subequivalence relations of E that are in the class $\mathcal{R}$. In particular $\mathcal{E}_E = S(E)$. Recall that an equivalence relation is **finite** if all its equivalence classes are finite and **hyperfinite** if it is the union of an increasing sequence of finite equivalence relations.

Theorem 8.1. *Let $\mathcal{H}$ be the class of hyperfinite equivalence relations. Then $\mathcal{H}_E$ is closed in $S(E)$.*

Proof. By Theorem 5.2. $\square$

Denote by $\mathcal{F}$, resp., $\mathcal{BF}$ the classes of equivalence relations which are **finite**, resp., **bounded finite** (i.e., for some N each equivalence class has at most N elements). It follows that

$$\overline{\mathcal{F}_E} = \overline{\mathcal{BF}_E} = \mathcal{H}_E.$$

In particular, E is hyperfinite iff $\mathcal{F}_E$ is dense in $S(E)$ iff $\mathcal{BF}_E$ is dense in $S(E)$. It also follows from this that the map $E \in S(F) \mapsto \mathcal{H}_E \in \mathcal{F}^*(S(F))$ is Borel (when $\mathcal{F}^*(S(F))$ is equipped with the Effros Borel structure). To see this let $\{F_n \colon n \in \mathbb{N}\}$ be a countable dense subset of $\mathcal{F}_F$ and let $\Phi \colon S(F) \to S(F)^{\mathbb{N}}$ be given by $\Phi(E)_n = E \cap F_n$. Then Φ is Borel and for each E it gives a dense sequence in $\mathcal{F}_E$, and so in $\mathcal{H}_E$, thus the map $E \in S(F) \mapsto \mathcal{H}_E \in \mathcal{F}^*(S(F))$ is Borel.

Next we calculate the complexity of the class of aperiodic equivalence relations.

Theorem 8.2. *Let $\mathcal{A}$ be the class of aperiodic equivalence relations. Then $\mathcal{A}_E$ is a G_δ set in the topology of $S(E)$. Moreover, if E is aperiodic, then $\mathcal{A}_E$ is dense.*

Proof. Let $\{T_n\} \subseteq [E]$ be a sequence of aperiodic automorphisms which is dense in the set of aperiodic elements of $[E]$. We claim that the following are equivalent for $F \in S(E)$:

(1) F is aperiodic,

(2) $\forall \epsilon \in \mathbb{Q}^+ \exists n (d(T_n, F) < \epsilon)$,

which clearly shows that the class of aperiodic elements of $S(E)$ is G_δ.

(1) $\Rightarrow$ (2). By [K, 3.5], $[F]$ contains an aperiodic T. Then for each $\epsilon \in \mathbb{Q}^+$ there is n such that $d(T_n, T) < \epsilon$, so $d(T_n, [F]) < \epsilon$.

(2) $\Rightarrow$ (1). Assume (2) and also that (1) fails, towards a contradiction. Then there is $N \in \mathbb{N}^+$ such that if $A = \{x \colon |[x]_F| = N\}$, then $\mu(A) = a > 0$.

Choose n so that $d(T_n, F) < \frac{a}{2(N+1)^2}$ and let $T \in [F]$ be such that $d(T_n, T) < \frac{a}{2(N+1)^2}$. Now note that for $i \leq N$, $d(T_n^i, T^i) \leq \frac{ia}{2(N+1)^2} \leq \frac{(N+1)a}{2(N+1)^2}$, so $\mu(\{x \colon \exists i \leq N (T^i(x) \neq T_n^i(x))\} \leq \frac{(N+1)^2 a}{2(N+1)^2} = \frac{a}{2}$. Therefore if $B = \{x \colon \forall i \leq N (T^i(x) = T_n^i(x))\}$, then $\mu(B) \geq 1 - \frac{a}{2}$, so $\mu(A \cap B) > 0$.

If $x \in A \cap B$, then $T^i(x) = T_n^i(x)$, for $i \leq N$, so $N = |[x]_F| \geq |\{T^i(x) \colon i \leq N\}| = |\{T_n^i(x) \colon i \leq N\}| = N + 1$, a contradiction.

Finally we prove that $\mathcal{A}_E$ is dense in $S(E)$, if E is aperiodic. For that it is enough to show that if $F \in S(E)$ is finite, then there is a sequence $F_0 \supseteq F_1 \supseteq \dots$ with $F_n \in \mathcal{A}_E$ and $\bigcap_n F_n = F$. Let Y be a Borel transversal for F.

Note now that if R is an aperiodic equivalence relation, then there is a sequence of aperiodic $R \supseteq R_0 \supseteq R_1 \supseteq \dots$ such that $\bigcap R_n = id$ (the equality equivalence relation). To see this, let $T \in [R]$ be aperiodic and let R_n be the equivalence relation generated by T^{2^n}, $n = 0, 1, 2, \dots$. Apply this now to $R = E|Y$ to find R_n as above and let $F_n = R_n \vee F$. $\square$

We also have the following calculation concerning the Marker Lemma (see [KM, Lemma 6.7]).

Proposition 8.3. *There is a Borel function $\Xi \colon \mathcal{A}_E \to \mathrm{MALG}^{\mathbb{N}}$ such that for each $F \in \mathcal{A}_E$, $\Xi(F)_0 \supseteq \Xi(F)_1 \supseteq \dots$, $\mu(\Xi(F)_n) \to 0$ and each $\Xi(F)_n$ is a complete section of F.*

Proof. Use Proposition 4.18 and the proof of the Marker Lemma as in [KM, Lemma 6.7]. $\square$

It is clear that $\mathcal{BF}_E$ is F_σ in the topology of $S(E)$ and it is dense if E is hyperfinite. Moreover if E is aperiodic, so that $S(E) \setminus \mathcal{BF}_E$ is dense by Theorem 8.2, it follows that $\mathcal{BF}_E$ is in $F_\sigma \setminus G_\delta$ in the topology of $S(E)$ for E aperiodic, hyperfinite. From Theorem 6.1, if $E \subseteq F$, then the topology on $S(E)$ is the relative topology it inherits from $S(F)$. Since every aperiodic E contains an aperiodic, hyperfinite subequivalence relation, we have the following:

Theorem 8.4. *For every aperiodic E, $\mathcal{BF}_E$ is in $F_\sigma \setminus G_\delta$ and $\mathcal{A}_E$ is in $G_\delta \setminus F_\sigma$ in the topology of $S(E)$.*

Theorem 8.5. *The set $\mathcal{F}_E$ of finite equivalence relations in $S(E)$ is $F_{\sigma\delta}$ in the topology of $S(E)$.*

Proof. The proof is a variation of that of Theorem 8.2. Since every equivalence relation is included in an aperiodic one, by the paragraph preceding Theorem 8.4, we can assume that E is aperiodic.

First note that for each open set $V \subseteq [E]$, the set

$$\{F \in S[E] \colon [F] \cap V \neq \emptyset\}$$

is open in the weak topology of $S(E)$. To see this, let

$$V = \bigcup_n \{T \in [E] \colon d(T, T_n) < \epsilon_n\},$$

for some sequence $\{T_n\} \in [E]^{\mathbb{N}}$ and sequence (ϵ_n) of positive reals. Then $[F] \cap V \neq \emptyset \iff \exists n (d(T_n, F) < \epsilon_n)$, so the above set is clearly open.

Below let $A(E)$ be the set of aperiodic elements of $[E]$. We claim that the following are equivalent for $F \in S(E)$:

(1) F is not finite,

(2) $\exists a \in \mathbb{Q}^+ \forall N \in \mathbb{N}^+ \exists S \in A(E)$
$\exists T \in [F] \big[\mu(\{x \colon \forall i \leq N(S^i(x) = T^i(x))\}) > a\big]$.

Granting this, it is enough to see that the expression in the second line of (2) above defines an open set of F's (for each fixed S). Let

$$V = \{T \in [E] \colon \mu(\{x \colon \forall i \leq N(S^i(x) = T^i(x))\}) > a\}.$$

Clearly $V \subseteq [E]$ is open and this expression is equivalent to $[F] \cap V \neq \emptyset$, which by the above defines an open set of F's.

We finally prove the equivalence of (1) and (2).

(1) $\Rightarrow$ (2). Assume that F is not finite. Let A be an F-invariant Borel set of positive measure with $F|A$ aperiodic and let $a \in \mathbb{Q}^+$ be such that $\mu(A) > a$. Let $T_0 \in [F|A]$ be aperiodic and let $T \in [F]$ be such that $T|A = T_0|A$. Let also $S \in A(E)$ be such that $T|A = S|A$. Then for each $N \in \mathbb{N}^+, x \in A$, we have that $S^i(x) = T^i(x), \forall i \leq N$, so $\mu(\{x \colon \forall i \leq N(S^i(x) = T^i(x))\}) > a$.

(2) $\Rightarrow$ (1). Assume that (2) is true and fix $a \in \mathbb{Q}^+$ witnessing that. If (1) fails, towards a contradiction, find $N \in \mathbb{N}^+$ large enough so that $\mu(\{x \colon |[x]_F| \leq N\}) > 1 - a$. Then find $S \in A(E), T \in [F]$ so that $\mu(\{x \colon \forall i \leq N(S^i(x) = T^i(x))\}) > a$. Thus there is x so that $|[x]_F| \leq N$ but $S^i(x) = T^i(x)$, for $i \leq N$. Then $N \geq |[x]_F| \geq |\{T^i(x) \colon i \leq N\}| = |\{S^i(x) \colon i \leq N\}| = N + 1$, a contradiction. □

In an earlier version of this work, the following question was asked:

If E is aperiodic, is $\mathcal{F}_E$ in $F_{\sigma\delta} \setminus G_{\delta\sigma}$ for the topology of $S(E)$?

The following then provided an affirmative answer when E is ergodic.

Theorem 8.6. *If E is ergodic, then $\mathcal{F}_E$ is in $F_{\sigma\delta} \setminus G_{\delta\sigma}$ in the topology of $S(E)$.*

Proof. First notice that for any aperiodic E, $\mathcal{F}_E$ is not G_δ in the topology of $S(E)$. To see this, we can assume, by the paragraph preceding Theorem 8.4, that E is aperiodic, hyperfinite. In this case $\mathcal{F}_E$ is dense and disjoint from the dense G_δ set $\mathcal{A}_E$, so it cannot be G_δ. It follows (see [K2, 21.18] and proof of 22.10) that for any aperiodic E, $\mathcal{F}_E$ is F_σ-hard, i.e., for each F_σ subset $A \subseteq Y$, Y a zero-dimensional Polish space, there is a continuous function $f \colon Y \to S(E)$ such that $y \in A \iff f(y) \in \mathcal{F}_E$.

Since every ergodic E contains an ergodic, hyperfinite subequivalence relation, we can assume as before that E is ergodic, hyperfinite. Since every aperiodic, hyperfinite equivalence relation is contained in an ergodic, hyperfinite equivalence relation (see [K, Lemma 5.4]) and all ergodic, hyperfinite equivalence relations are isomorphic by Dye's Theorem, it is enough to find *some* aperiodic, hyperfinite equivalence relation E such that this theorem holds for E.

Given a sequence of measure preserving countable Borel equivalence relations (E_n) on (X,μ), define their **direct sum**, in symbols $\bigoplus_n E_n$, as follows: Let $Y = \bigsqcup_n X_n$ be the direct sum of infinitely many copies of X. On each X_n put a copy μ_n of the measure μ and define the measure ν on Y by $\nu = \sum_n \frac{1}{2^{n+1}}\mu_n$. Then put on each Y_n a copy E'_n of E_n, and let $\bigoplus_n E_n = \bigcup_n E'_n$. Clearly $\bigoplus_n E_n$ is a measure preserving equivalence relation on (Y,ν). Moreover the map $\prod_n S(E_n) \to S(\bigoplus_n E_n)$ given by $(F_n) \mapsto \bigoplus_n F_n$ is a homeomorphism of $\prod_n S(E_n)$ with $S(\bigoplus_n E_n)$, each equipped with the weak topology. Moreover, under this homeomorphism $\prod_n \mathcal{F}_{E_n}$ goes to $\mathcal{F}_{\bigoplus_n E_n}$.

Take now each E_n to be aperiodic, hyperfinite, so that $E = \bigoplus_n E_n$ is also aperiodic, hyperfinite. Then each $\mathcal{F}_{E_n}$ is F_σ-hard and so $\prod_n \mathcal{F}_{E_n}$ is $\mathcal{F}_{\sigma\delta}$-hard and thus $\mathcal{F}_E$ is also $\mathcal{F}_{\sigma\delta}$-hard, which completes the proof. □

Recently Le Maître, in [LeM1], gave a positive answer in general for *every* aperiodic E.

Theorem 8.7 (Le Maître, [LeM1]). *If E is aperiodic, then $\mathcal{F}_E$ is in $F_{\sigma\delta} \setminus G_{\delta\sigma}$ in the topology of $S(E)$.*

Let $\mathcal{T}$ be the class of treeable equivalence relations and let $\mathcal{D}_n, n = 1,2,\dots$, be the class of equivalence relations that have **geometric dimension** $\leq n$, i.e., can be Borel reduced (a.e.) to a $\mathcal{K}_n$-structurable Borel equivalence relation, where $\mathcal{K}_n$ is the class of n-dimensional contractible simplicial complexes (see Gaboriau [G1, 3.18], and Hjorth-Kechris [HK, Appendix D]). Thus $\mathcal{D}_1 = \mathcal{T}$. Gaboriau [G1, 5.8], shows that $\mathcal{D}_n$ is hereditary and by [G1, 3.17], if $E \in \mathcal{D}_n$, then $\beta_p(E) = 0$, if $p > n$, where β_p is the p-th L^2-Betti number. Recall also from [G1, 3.16], that if F is induced by a free measure preserving action of $(\mathbb{F}_2)^n$, then $\beta_n(F) = 1$.

Let $\mathcal{D}_{n,E} = (\mathcal{D}_n)_E$. We have $\mathcal{T}_E = \mathcal{D}_{1,E} \subsetneqq \mathcal{D}_{2,E} \subsetneqq \cdots \subsetneqq \mathcal{D}_{n,E} \subsetneqq \dots$, for any large enough E. This is because an equivalence relation F_n induced by a free measure preserving action of $(\mathbb{F}_2)^{n+1}$ is in $\mathcal{D}_{n+1} \setminus \mathcal{D}_n$, for $n \geq 1$. That $F_n \in \mathcal{D}_{n+1}$ follows from [G1, 5.17], and $F_n \notin \mathcal{D}_n$ since $\beta_{n+1}(F) = 1$.

Gaboriau [G1, 5.13], also shows that if $R_0 \subseteq R_1 \subseteq \dots$ are measure preserving countable Borel equivalence relations, with $R = \bigcup_i R_i$, then $\beta_n(R) \leq \underline{\lim}_i \beta_n(R_i)$, thus if all $\beta_n(R_i) = 0$, we also have that $\beta_n(R) = 0$. It follows that if $R_i \in \mathcal{D}_n$, so that $\beta_p(R_i) = 0$, for $p > n$, then $\beta_p(R) = 0$, if

$p > n$. Therefore if $R \in (\mathcal{D}_n)_\uparrow$, then $\beta_p(R) = 0$, if $p > n$. In particular, if E is large enough so that it contains equivalence relations induced by free measure preserving actions of $(\mathbb{F}_2)^n, n = 1, 2, \dots$, then no $\mathcal{D}_{n,E}$ is dense in $S(E)$.

Problem 8.8. *Let $\mathcal{D}_\infty = \bigcup_n \mathcal{D}_n$. Is $\mathcal{D}_{\infty,E}$ dense in the topology of $S(E)$ (for large enough E)?*

We also note that for $n \geq 1, (\mathcal{D}_n)_\uparrow \neq \mathcal{D}_n$, thus no $\mathcal{D}_{n,E}$ is closed in the topology of $S(E)$ (if E is large enough). To see this, let F_2 be the equivalence relation induced by the shift action of $\mathbb{F}_2$ and let F_1 be the equivalence relation induced by the shift action of $Z = \bigoplus_n (\mathbb{Z}/2)^n$. Then $F = (F_2)^n \times F_1$, is induced by a free measure preserving action of $(\mathbb{F}_2)^n \times Z$. Gaboriau (see [G2, 7.3]) has shown that the ergodic dimension of $(\mathbb{F}_2)^n \times Z$ is $n+1$. Recall that the **ergodic dimension** of a group is the minimum of the geometric dimensions of the equivalence relations given by free measure preserving actions of the group. It follows that the geometric dimension of F is $\geq n + 1$, thus $F \notin \mathcal{D}_n$. On the other hand it is easy to see (see, e.g., [HK, page 62]) that $F \in (\mathcal{D}_n)_\uparrow$.

Finally notice that by [G2, 7.3], (a), $(\mathcal{D}_n)_\uparrow \subseteq \mathcal{D}_{n+1}$, so for every E, $\mathcal{D}_{\infty,E} = \bigcup_n \mathcal{D}_{n,E} = \bigcup_n \overline{\mathcal{D}_{n,E}}$ is an F_σ set.

Problem 8.9. *What is the descriptive complexity of each $\mathcal{D}_{n,E}, n \geq 1$, in the topology of $S(E)$ (for large enough E)? Is $\mathcal{D}_{\infty,E}$ a true F_σ set?*

We will see later in Corollary 19.5 that $\mathcal{T}_E$ is analytic in $S(E)$ but it is not known if it is Borel; see Problem 19.6.

9. Finite and infinite index subrelations

Denote by $FinIndex(E)$ (resp., $InfIndex(E)$) the set of all $F \in S(E)$ such that $[E : F] < \infty$, i.e., every E-class contains only finitely many F-classes (resp., $[E : F] = \infty$, i.e., every E-class contains infinitely many F-classes).

Proposition 9.1. *The set $InfIndex(E)$ is G_δ in $S(E)$ and it is dense if E is aperiodic.*

Proof. Let (T_i) be a generating sequence for E. Then the following are equivalent for $F \in S(E)$:

(i) $F \in InfIndex(E)$,
(ii) $\forall n \forall k > 0 \exists M(\mu(\{x \colon \exists m \leq M \forall i \leq n \neg T_m(x) F T_i(x)\}) > 1 - \frac{1}{k})$.

Let

$$B_{F,M,n} = \bigcup_{m \leq M} \bigcap_{i \leq n} (X \setminus T_i^{-1}(A_{T_m T_i^{-1}, F})).$$

Then

$$InfIndex(E) = \bigcap_n \bigcap_{k>0} \bigcup_M \{F \colon \mu(B_{F,M,n}) > 1 - \frac{1}{k}\}.$$

Since $F \mapsto \mu(B_{F,M,n})$ is continuous, this shows that $InfIndex(E)$ is G_δ.

Assume now that E is aperiodic and let $A_0 \supseteq A_1 \supseteq \dots$ be Borel sets which are complete sections of E and $\mu(A_n) \to 0$. Given $F \in S(E)$, let $F_n = F|(X \setminus A_n) \sqcup id|A_n$. Then (F_n) is increasing and $F = \bigcup F_n$, so $F_n \to F$. Since each A_n meets every E-class in an infinite set, there are infinitely many F_n-classes in each E-class, i.e., $F_n \in InfIndex(E)$. $\square$

Remark 9.2. A similar calculation gives another way to show that $\mathcal{A}_E$ is G_δ is $S(E)$ (see Theorem 8.2). Indeed as in the proof of Proposition 9.1, put

$C_{i,m} = \{x\colon T_i(x) \neq T_m(x)\}$ and $D_{F,M,n} = \bigcup_{m\leq M} \bigcap_{i\leq n}(C_{i,m} \cap A_{F,T_m})$. Then $\mathcal{A}_E = \bigcap_n \bigcap_{k>0} \bigcup_M \{F\colon \mu(D_{F,M,n}) > 1 - \frac{1}{k}\}$ and since $F \mapsto D_{F,M,n}$ is again continuous, $\mathcal{A}_E$ is G_δ.

Answering a question raised in an earlier version of this work, Le Maître in [LeM1] showed the following:

Theorem 9.3 (Le Maître, [LeM1])**.** *If E is aperiodic, then $InfIndex(E)$ is $G_\delta \setminus F_\sigma$ in the topology of $S(E)$.*

Proposition 9.4. *The set $FinIndex(E)$ is $F_{\sigma\delta}$ in $S(E)$. If E is aperiodic, hyperfinite, then it is also dense.*

Proof. Let (T_i) be a generating sequence for E. Let

$$L_{F,M,n} = \{x\colon \forall i \leq n \exists m \leq M(T_i(x) F T_m(x))\}.$$

Then

$$FinIndex(E) = \bigcap_{k>0} \bigcup_M \bigcap_n \{F\colon \mu(L_{F,M,n}) \geq 1 - \frac{1}{k}\}$$

and since the map $F \mapsto L_{F,M,n}$ is continuous, this shows that $FinIndex(E)$ is $F_{\sigma\delta}$.

Assume now that E is aperiodic, hyperfinite. It is enough to approximate every smooth $F \in S(E)$ by finite index subrelations of E. Let Y be a Borel transversal for F. Then $E|Y$ is aperiodic (on Y), so there is aperiodic $S \in [E|Y]$ which generates $E|Y$. Let $F_n = F \vee E_{S^{2^n}}$ (note that $E_{S^{2^n}}$ is an equivalence relation on Y, which we can view as an equivalence relation on X but extending it by equality outside Y). Then (F_n) is decreasing, $F_n \in FinIndex(E)$ and $F_n \to F$. $\square$

Again answering a question raised in an earlier version of this work, Le Maître in [LeM1] showed the following:

Theorem 9.5 (Le Maître, [LeM1])**.** *If E is aperiodic, then $FinIndex(E)$ is $F_{\sigma\delta} \setminus G_{\delta\sigma}$ (in the topology of $S(E)$) iff E has infinitely many ergodic components. Otherwise it is in $F_\sigma \setminus G_\delta$.*

We next show that $FinIndex(E)$ is not always dense in $S(E)$.

Theorem 9.6. *There is an ergodic E such that $FinIndex(E)$ is not dense in $S(E)$.*

Proof. Let Γ be an infinite property (T) group all of whose proper subgroups are finite (such groups exist by a result of Olshanskii, see [DC, Proposition 2] and [O, Corollary 4]). Consider the shift action of Γ on $X = [0,1]^\Gamma$ and denote by E the associated equivalence relation. We will show that this works.

Call $F \in FinIndex(E)$ **degenerate** if there is a Borel partition $X = A_0 \sqcup A_1 \sqcup \cdots \sqcup A_{n-1}$ into sets of positive measure such that

$$xFy \iff xEy \;\&\; \exists i < n(x, y \in A_i).$$

Such an F is denoted by $E_{A_0,A_1,\dots A_{n-1}}$.

The next fact strengthens the last part of Bowen [Bo1, Theorem 1.1].

Lemma 9.7. *If $F \in FinIndex(E)$, then F is degenerate.*

Proof. Let $[E : F] = n$ be the **index** of F in E, i.e., the number of F-classes in each E-class. Let also $(\varphi_i)_{i<n}$ be a **choice sequence** for F in E, i.e., a sequence of Borel functions such that for each x, $([\varphi_i(x)]_F)_{i<n}$ is an injective enumeration of the F-classes in $[x]_E$. Let also $\sigma : E \to S_n$ (= the symmetric group of n elements) be the associated **index cocycle** defined by

$$\sigma(x, y)(i) = j \iff \varphi_i(x) F \varphi_j(y).$$

This of course can also viewed as a cocycle of the shift action of Γ into S_n, so by Popa superrigidity, see [Po], it is cohomologous to a homomorphism from Γ into S_n, which, since Γ has no proper finite index subgroups, must be trivial, i.e., σ is a coboundary and so by [FSZ, Proposition 1.7], F is degenerate. □

Call $F \in S(E)$ **relatively smooth**, resp., **relatively hypersmooth**, if there is a smooth (resp., hypersmooth) Borel equivalence relation R such that $F = E \cap R$.

Lemma 9.8. *If $F \in S(E)$ is the limit of a sequence of degenerate relations, then F is relatively hypersmooth.*

Proof. Let $F_i = E_{A^i_0,\dots,A^i_{n_i-1}}$ be such that $F_i \to F$. Then by Theorem 5.1, for each i, there is an increasing sequence $n^{(i)}_0 < n^{(i)}_1 < \dots$, so that $(n^{(i+1)}_m)_{m\in\mathbb{N}}$ is a subsequence of $(n^{(i)}_m)_{m\in\mathbb{N}}$ and

$$F = \bigcup_m \bigcap_{k\geq m} F_{n^{(m)}_k}.$$

Put $R_m = \bigcap_{k \geq m} F_{n_k^{(m)}}$. Then $R_0 \subseteq R_1 \dots$ and $F = \bigcup_m R_m$. Define for each m,

$$f_m\colon X \to \mathbb{N}^{\mathbb{N}}$$

by

$$f_m(x)(i) = n \iff x \in A_n^{n_{m+i}^{(m)}}.$$

Then if

$$xS_m y \iff f_m(x) = f_m(y),$$

we have $R_m = E \cap S_m$. Also $S_0 \subseteq S_1 \subseteq \dots$ and $F = \bigcup_m R_m = E \cap (\bigcup_m S_m)$ and $\bigcup_m S_m$ is hypersmooth. □

By a result of Gaboriau-Lyons [GL], there is a free, measure preserving, ergodic action of $\mathbb{F}_2$ whose induced equivalence relation F is in $S(E)$. Then by the result of Chifan-Ioana [CI], F is strongly ergodic (see Section 10.2 for the definition of strong ergodicity). We claim that F cannot be the limit of a sequence of degenerate relations, thus it is not in the closure of $FinIndex(E)$. Otherwise, by Lemma 9.8, we would have $F = E \cap R$, with R hypersmooth, say $R = \bigcup_n R_n$, with (R_n) increasing and each R_n smooth. Let $f_n\colon X \to 2^{\mathbb{N}}$ be Borel such that $xR_n y \iff f_n(x) = f_n(y)$. Let $F_n = E \cap R_n$, so that (F_n) is increasing and $F = \bigcup_n F_n$. By a result of Gaboriau [G2, Proposition 5.2], there is n and a F_n-invariant Borel set A of positive measure such that $F_n|A$ is ergodic. Since $f_n|A$ is $F_n|A$-invariant, it is constant, so $F_n|A = E|A$ and thus

$$F_n|A \subseteq F|A \subseteq E|A = F_n|A,$$

i.e, $F|A = E|A$. But $F|A$ is treeable, so $E|A$ is treeable and, since A is a complete section for E, E is treeable, contradicting the result of Adams and Spatzier [AS]. □

Remark 9.9. For an arbitrary E, it is the case that $F \in S(E)$ is relatively hypersmooth iff F is the limit of a sequence of degenerate relations. One direction is proved as in Lemma 9.8 (which did not use any particular properties of E). For the other direction it is enough to show that every relatively smooth $F \in S(E)$ is the limit of degenerate relations. Indeed. let R be smooth such that $F = E \cap R$ and let $f\colon X \to 2^{\mathbb{N}}$ be a Borel function such that $xRy \iff f(x) = f(y)$. For $s \in 2^n, n \in \mathbb{N} \setminus \{0\}$, let $N_s = \{x \in 2^{\mathbb{N}}\colon x|n = s\}$ and $A_s = f^{-1}(N_s)$. Consider then, for each $n > 0$, the degenerate relation F_n determined by the partition $\{A_s\}_{s\in 2^n}$. Clearly $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ and $F = \bigcap_n F_n$, so F is the limit of the sequence (F_n).

Problem 9.10. *For what ergodic E is $FinIndex(E)$ dense in $S(E)$?*

Remark 9.11. In Vaes [Va] and Bowen [Bo1] examples are given of ergodic equivalence relations that do not have proper finite index ergodic subequivalence relations or proper finite index extensions.

Remark 9.12. In Popa [Po, Section 6.6] it is suggested that it might be possible that the cocycle superrigidity proved in that paper could be extended to target groups that are closed subgroups of the (infinitary) unitary group $U(H)$. One can see however that this fails for the infinite symmetric group S_∞, which is a closed subgroup of $U(H)$. Indeed let Γ, E be as in the proof of Theorem 9.6. Let $F \in S(E)$ be ergodic, hyperfinite, so that $[E : F] = \infty$. Let $(\varphi_i)_{i<\infty}$ be a choice sequence for F in E and let σ be the associated index cocycle, which now takes values in S_∞. Assume, towards a contradiction, that this is cohomologous to a homomorphism $\pi\colon \Gamma \to S_\infty$. Thus there is a Borel map $p\colon X \to S_\infty$ such that

$$\sigma(x, \gamma\cdot x) = p(\gamma\cdot x)\pi(\gamma)p(x)^{-1}.$$

Put $\psi_i(x) = \varphi_{p(x)(i)}(x)$, so that (ψ_i) is also a choice sequence with associated index cocycle $\tau(x, \gamma\cdot x) = \pi(\gamma)$, so that $\psi_i(x) F \psi_{\pi(\gamma)(i)}(\gamma\cdot x)$. Since $\forall x \exists i(\psi_i(x) F x)$, fix i_0 such that if $A = \{x\colon \psi_{i_0}(x) F x\}$, then A has positive measure. By the ergodicity of F, A meets every F-class infinitely often.

Now if $x, \gamma\cdot x \in A$ and $xF\gamma\cdot x$, we have

$$xF\psi_{i_0}(x)F\psi_{\pi(\gamma)(i_0)}(\gamma\cdot x)F\gamma\cdot xF\psi_{i_0}(\gamma\cdot x)$$

so

$$\psi_{\pi(\gamma)(i_0)}(\gamma\cdot x)F\psi_{i_0}(\gamma\cdot x),$$

thus $\pi(\gamma)(i_0) = i_0$. It follows that

$$\Delta = \{\gamma\colon \pi(\gamma)(i_0) = i_0\}$$

is an infinite subgroup of Γ, so $\Delta = \Gamma$, i.e., $\forall\gamma \in \Gamma(\pi(\gamma)(i_0) = i_0)$. Then

$$\psi_{i_0}(x)F\psi_{\pi(\gamma)(i_0)}(\gamma\cdot x) = \psi_{i_0}(\gamma\cdot x),$$

so $x \mapsto \psi_{i_0}(x)$ is a homomorphism of E into F. Since E is strongly ergodic and F is hyperfinite, this maps a.e. to a single F-class, which is a contradiction, since $\psi_{i_0}(x)Ex$.

More generally, one can show that if E is induced by a free, measure preserving, ergodic action of a countable infinite group Γ on a standard measure space (X, μ), if $F \in S(E)$ is aperiodic and the index cocycle of F in E is cohomologous to a homomorphism, then there is a Borel decomposition $X = \bigsqcup_n X_n$ and infinite subgroups Δ_n of Γ such that if E_n is the equivalence relation induced by the restriction of the action to Δ_n, then $E_n|X_n = F|X_n$.

10. Ergodic and strongly ergodic equivalence relations

We discuss here the complexity of the notions of ergodicity and strong ergodicity.

10.1 Ergodic equivalence relations

We first calculate the complexity of the set of ergodic equivalence relations in $S(E)$. We denote by $\mathcal{ERG}$ the class of (measure preserving countable Borel) equivalence relations which are ergodic.

Theorem 10.1. *The set $\mathcal{ERG}_E$ of ergodic equivalence relations in $S(E)$ is G_δ in $S(E)$.*

Proof. We will give two proofs based, resp., in two descriptions of the topology of $S(E)$ given in Section 4.4.

(1) (with R. Tucker-Drob) In the notation of Section 4.4, (2) we have the following fact:

Lemma 10.2. *The set* $\mathrm{ERG} = \{\mu \in \mathcal{M} \colon \mu$ *is R-invariant, ergodic*$\}$ *is G_δ in $\mathcal{M}$.*

Proof. The set $\{\mu \in \mathcal{M} \colon \mu$ is R-invariant$\}$ is compact, convex and, since R is a countable Borel equivalence relation, the ergodic measures in $\{\mu \in \mathcal{M} \colon \mu$ is R-invariant$\}$ are exactly its extreme points, which clearly form a G_δ set. $\square$

Then

$$\mathcal{ERG}_E = \Phi^{-1}(\mathrm{ERG})$$

and since $\Phi \colon S(E) \to \mathcal{M}$ is continuous, $\mathcal{ERG}_E$ is G_δ in $S(E)$.

(2) (P. Burton) In the notation of Section 4.4, (3), we note that if we let $\mathrm{ERG}(\mathbb{F}_\infty, X, \mu)$ be the set of ergodic actions in $A(\mathbb{F}_\infty, X, \mu)$, then we have that $\mathrm{ERG}(\mathbb{F}_\infty, X, \mu)$ is G_δ in the weak topology of $A(\mathbb{F}_\infty, X, \mu)$ (see [K, Proposition 12.1]) and thus it is also G_δ in the uniform topology. Since Ψ is a homeomorphism between $S(E)$ and a closed subspace of $A(\mathbb{F}_\infty, E)$ with the uniform topology and for $F \in S(E)$, $F = E_{\Psi(F)}$, we have that $\mathcal{ERG}_E = \Psi^{-1}(\mathrm{ERG}(\mathbb{F}_\infty, X, \mu))$, so $\mathcal{ERG}_E$ is G_δ in $S(E)$. $\square$

10.2 Strongly ergodic equivalence relations

An equivalence relation F is called **strongly ergodic** or $\boldsymbol{E}_0$**-ergodic** iff for any Borel **homomorphism** $\pi\colon X \to Y$ from F to a hyperfinite equivalence relation R on Y (i.e., $xFx' \implies \pi(x)R\pi(x')$), there is $y \in Y$ such that $\pi^{-1}([y]_F)$ has measure 1. By a result of Jones-Schmidt this is equivalent to the non-existence of non-trivial almost invariant sets for F (see, e.g., [HK, Theorem A2.2], in which the hypothesis of ergodicity is unnecessary). We denote the class of all (measure preserving countable Borel) equivalence relations that are strongly ergodic by $\mathcal{E}_0\mathcal{RG}$.

We call an equivalence relation F **anti-**$\boldsymbol{E}_0$**-ergodic** if there is homomorphism π as above to a hyperfinite equivalence relation for which all preimages of F-classes are null. Denote by $\mathcal{AE}_0\mathcal{RG}$ the class of all anti-E_0-ergodic equivalence relations.

Proposition 10.3. *The set $\mathcal{AE}_0\mathcal{RG}_E$ is closed in $S(E)$.*

Proof. Miller [M, 2.1], has shown that $\mathcal{AE}_0\mathcal{RG}_E$ is closed under taking unions of increasing sequences. It is obvious that it is also hereditary, so by Theorem 5.2 it is closed. $\square$

Theorem 10.4. *The set $\mathcal{E}_0\mathcal{RG}_E$ is in the class $F_\sigma \cap G_\delta$ in $S(E)$.*

Proof. Consider $\mathcal{S} = S(E) \setminus \mathcal{E}_0\mathcal{RG}_E$. Then clearly $\mathcal{ERG}_E \cap \mathcal{S} = \mathcal{ERG}_E \cap \mathcal{AE}_0\mathcal{RG}_E$. Moreover $S(E) \setminus \mathcal{ERG}_E \subseteq \mathcal{S}$. Thus $\mathcal{S} = (\mathcal{ERG}_E \cap \mathcal{AE}_0\mathcal{RG}_E) \sqcup (S(E) \setminus \mathcal{ERG}_E)$, which is in $G_\delta \cup F_\sigma$ by Theorem 10.1 and Proposition 10.3. $\square$

Problem 10.5. *Are there E for which Theorem 10.4 gives the optimal descriptive complexity of $\mathcal{E}_0\mathcal{RG}_E$?*

11. Richly ergodic equivalence relations

We first note that for any E, $S(E) \setminus \mathcal{ERG}_E$ is dense in $S(E)$. This is because any $F \in S(E)$ can be approximated by equivalence relations of the form $F|(X \setminus A) \sqcup id|A$, for Borel A of small positive measure, which clearly are not ergodic.

We discuss here the following problem:

Problem 11.1. *For which ergodic equivalence relations E is the set $\mathcal{ERG}_E$ dense in $S(E)$?*

Let us call an ergodic equivalence relation E for which $\mathcal{ERG}_E$ is dense in $S(E)$ **richly ergodic**. We first show that there exist ergodic but not richly ergodic equivalence relations. These arise in the context of the so-called non-approximable equivalence relations, introduced in the paper Gaboriau–Tucker-Drob [GT].

Definition 11.2. *Let E be a measure preserving countable Borel equivalence relation on (X, μ). We say that E is* **non-approximable** *if whenever $E = \bigcup_n F_n$, where F_n are Borel equivalence relations with $F_0 \subseteq F_1 \subseteq F_2 \dots$, then there is n and a positive measure Borel set A with $E|A = F_n|A$.*

It is an unpublished result of Gaboriau that if $a \in A(\Gamma, X, \mu)$, where Γ is an infinite property (T) group, and a is ergodic, then the equivalence relation E_a is non-approximable. This can be also seen as an application of [IKT, Corollary 5.4 and Corollary 2.15]. In [GT] the authors also show that if $a \in A(\Gamma \times \Delta, X, \mu)$ is a free action, where Γ, Δ are finitely generated, and $a|\Gamma$ is strongly ergodic while $a|\Delta$ is ergodic, then E_a is non-approximable. We now have:

Proposition 11.3. *If E is ergodic and non-approximable, then E is not richly ergodic.*

Proof. First we show that E is an isolated point in $\mathcal{ERG}_E$. Otherwise there is a sequence $F_n \in \mathcal{ERG}_E$ such that $F_n \to E$ and $F_n \neq E, \forall n$. By Theorem 5.1, we can write $E = \bigcup_m R_m$, with $R_0 \subseteq R_1 \subseteq \dots$, and for each m, there is n such that $R_m \subseteq F_n$. Since E is non-approximable, there is m, n and a positive measure Borel set A such that $E|A = R_m|A \subseteq F_n|A \subseteq E|A$, so that $E|A = F_n|A$. Since E is ergodic, A is a complete section for E. Let $B = [A]_{F_n}$. Then we also have $E|B = F_n|B$. Since B has positive measure, is F_n-invariant and F_n is ergodic, $B = X$ (modulo null sets) and so $E = F_n$, a contradiction.

If now E was richly ergodic, it would follow that E is also an isolated point in $S(E)$. However it is easy to see that $S(E)$ is perfect, i.e., has no isolated points. This follows from the remarks in the first paragraph of this section. □

We will next see that in some sense most E are richly ergodic (see the paragraph following Problem 6.11 here). If $E_0 \subseteq E_1 \subseteq \dots$ is an increasing sequence, we say that $(E_n)_{n\in\mathbb{N}}$ is **strongly increasing** if for each n there is an ergodic $T \in [\bigcup_n E_n]$ such that $E_n \perp E_T$.

Proposition 11.4. *If (E_n) is strongly increasing and $E = \bigcup_n E_n$, then E is richly ergodic.*

Proof. Let $F \in S(E)$. Then $F \cap E_n \to F$, so it is enough to show that for each n, $S(E_n)$ is contained in the closure of $\mathcal{ERG}_E$. Fix $F \in S(E_n)$. Let then $T \in [E]$ be ergodic with $E_T \perp E_n$. By Dye's Theorem, there is $S \in [E_T]$, S mixing. Then also $F \perp E_S$. Put $F_n = F \vee E_{S^{2^n}} \in S(E)$. Then (F_n) is decreasing, $\bigcap_n F_n = F$, so that $F_n \to F$, and each F_n is ergodic. □

Proposition 11.5. *For any E, there is $E' \supseteq E$ which is richly ergodic.*

Proof. Recall that for each equivalence relation R, the set

$$\{T \in \mathrm{Aut}(X, \mu) \colon E_T \perp R\}$$

is comeager in the weak topology of $\mathrm{Aut}(X, \mu)$ (see Conley-Miller [CM, Theorem 8]). Since the set of ergodic automorphisms in $\mathrm{Aut}(X, \mu)$ is also comeager, it follows that there is an ergodic T with $R \perp E_T$.

Define now recursively $E_0 \subseteq E_1 \subseteq \dots$, by $E_0 = E, E_{n+1} = E_n \vee E_{T_n}$, where $E_{T_n} \perp E_n$ and T_n is ergodic. Then (E_n) is strongly increasing and thus $E' = \bigcup_n E_n$ is richly ergodic. $\square$

Proposition 11.6. *If $E_0 \subseteq E_1 \subseteq \dots$ are richly ergodic, so is $E = \bigcup_n E_n$.*

Proof. If $F \in S(E)$, then $F \cap E_n \to F$ and each $F \cap E_n$ is the limit of ergodic equivalence relations contained in E_n. $\square$

Thus the collection of richly ergodic equivalence relations is ω-closed and cofinal in the class of all equivalence relations. We next discuss some classes of richly ergodic equivalence relations. Below let $\mathcal{ERGH} = \mathcal{ERG} \cap \mathcal{H}$ be the class of ergodic, hyperfinite equivalence relations.

Proposition 11.7. *For any ergodic E, $\mathcal{ERGH}_E$ is dense in $\mathcal{H}_E$. In particular, every hyperfinite ergodic equivalence relation is richly ergodic.*

Proof. It is enough to show that if $F \in S(E)$ is smooth, then F is the limit of ergodic, hyperfinite equivalence relations in $S(E)$.

Let Y be a Borel transversal for F. Then $E|Y$ is ergodic (on Y), so there is $S \in [E|Y]$ which is mixing. Let $F_n = F \vee E_{S^{2^n}}$ (note that $E_{S^{2^n}}$ is an equivalence relation on Y, which we can view as an equivalence relation on X but extending it by equality outside Y). Then (F_n) is decreasing and each F_n is ergodic. Indeed, if a Borel set A is F_n-invariant, then $A \cap Y$ is $E_{S^{2^n}}$-invariant, so, since $E_{S^{2^n}}$ is ergodic (on Y), we have that $\mu(A \cap Y) = 0$ or $\mu(Y \setminus A) = 0$, thus $\mu(A) = 0$ or $\mu(X \setminus A) = 0$, since $A = [A \cap Y]_F$ and similarly for $X \setminus A$. Moreover Y is a complete section of F_n and $F_n|Y = E_{S^{2^n}}$ is hyperfinite, so F_n is hyperfinite.

Finally we claim that $\bigcap_n F_n = F$, which completes the proof. Let $p\colon X \to Y$ be the Borel selector corresponding to Y, i.e., $p(x) \in Y$ and $xFp(x)$. Then if $(x, y) \in \bigcap_n F_n$, we have that for each n there is a unique $k_n \in \mathbb{Z}$ with $p(y) = S^{k_n 2^n}(p(x))$. Since S is aperiodic, this can only happen if $p(x) = p(y)$, i.e., xFy. $\square$

Proposition 11.8. *Let $\Gamma = \Gamma_1 * \Gamma_2 * \cdots$, where each countable group Γ_n is non-trivial. Let E be induced by a free, measure preserving, mixing action of Γ. Then E is richly ergodic.*

Proof. Let $\Delta_n = \Gamma_1 * \cdots * \Gamma_n$ and let E_n be the equivalence relation induced by the restriction of the action to Δ_n. Then (E_n) is clearly increasing with

$\bigcup_n E_n = E$ and we claim that (E_n) is strongly increasing. This is because $\Gamma_{n+1} * \Gamma_{n+2}$ contains an element of infinite order, say δ. If $T \in [E]$ corresponds to the action of δ, then T is ergodic and clearly $E_n \perp E_T$. So, by Proposition 11.4, E is richly ergodic. □

Finally we call an equivalence relation E **richly E_0-ergodic** if $\mathcal{E}_0\mathcal{RG}_E$ is dense in $S(E)$.

Problem 11.9. *Which E_0-ergodic equivalence relations E are richly E_0-ergodic?*

By Proposition 11.3 and the paragraph preceding it, it clearly follows that there are E_0-ergodic equivalence relations which are not richly E_0-ergodic. There are also richly E_0-ergodic equivalence relations, One way to see this is by using a variation of the construction in Proposition 11.8.

Let $\Gamma = \mathbb{F}_\infty = \langle \gamma_0, \gamma_1, \dots \rangle$. Let a be the shift action of Γ on 2^Γ, equipped with the usual product measure. Then for any non-amenable $\Delta \leq \Gamma$, the restriction $a|\Delta$ of this action to Δ is E_0-ergodic (see, e.g., [HK, Theorem A4.1]). Let $E = E_a$, $\Gamma_n = \langle \gamma_0, \gamma_1, \dots, \gamma_n \rangle$, $a_n = a|\Gamma_n$ and $E_n = E_{a_n}$. Then the E_n are increasing and $E = \bigcup_n E_n$. We will check that E is richly E_0-ergodic. For this it is enough to show that for each n and $F \in S(E_n)$, F is the limit of E_0-ergodic equivalence relations in $S(E)$.

Let $\Delta_m \leq \langle \gamma_{n+1}, \gamma_{n+2} \rangle$ be non-abelian subgroups with $\Delta_0 \supseteq \Delta_1 \dots$ and $\bigcap_m \Delta_m = \{1\}$. Put $b_m = a|\Delta_m$ and $R_m = E_{b_m}$, so that R_m is E_0-ergodic. Also clearly $R_m \perp F$. Let $F_m = F \vee R_m$. Then F_m is E_0-ergodic, $F_0 \supseteq F_1 \dots$ and $\bigcap_m F_m = F$, so $F_m \to F$.

12. The cost function

For an equivalence relation F denote by $C(F) = C_\mu(F)$ the **cost** of F. We will discuss here the complexity of the function $F \in S(E) \mapsto C(F) \in [0, \infty]$.

Let

$$FinCost_E = \{F \in S(E) \colon C(F) < \infty\},$$
$$InfCost_E = \{F \in S(E) \colon C(F) = \infty\}.$$

Proposition 12.1. *The set $FinCost_E$ is dense in $S(E)$.*

Proof. Let $F \in S(E)$ and fix $a \in A(\mathbb{F}_\infty, X, \mu)$ with $F = E_a$. Let $F_n = E_{a|\mathbb{F}_n}$. Then $F_n \to F$ and each F_n has finite cost. □

We next have the following dichotomy:

Theorem 12.2. *For any aperiodic equivalence relation E, exactly one of the following holds:*

(i) For every $F \in S(E)$, $C(F) \leq 1$,

(ii) $InfCost_E$ is dense in the uniform topology of $S(E)$.

Proof. We will need the following lemma:

Lemma 12.3. *Let F be an equivalence relation with $C(F) > 1$. Then there is a subequivalence relation $F' \subseteq F$ with $C(F') = \infty$.*

Proof. We use the ideas in the proof of [KM, Proposition 28.8]. Consider the ergodic decomposition $\pi \colon X \to \mathcal{EI}_F$, where $\mathcal{EI}_F$ is the standard Borel space of F-ergodic invariant probability measures on X (we view here F as a genuine countable Borel equivalence relation and not one defined μ-a.e.); see Theorem 4.19. Let $\nu = \pi_*\mu$.

Put

$$Y_1 = \{e \in \mathcal{EI}_F \colon \ : C_e(F|X_e) > 1\}, Y_0 = \mathcal{EI}_F \setminus Y_1,$$

where $X_e = \pi^{-1}(\{e\})$. Then Y_1 is coanalytic and by [KM, Theorem 18.6] $\nu(Y_1) > 0$, so there is Borel $Z_1 \subseteq Y_1$ such that $\nu(Y_1) > 0$. Put $X_1 = \pi^{-1}(Z_1), X_0 = X \setminus X_1$. Then X_1 is Borel, $\mu(X_1) > 0$ and if $X_e \subseteq X_1$, then $C_e(F|X_e) > 1$. By the proof of [KM, Proposition 28.8], there is a free Borel action a of $\mathbb{F}_2$ on X_1 with $E_a \subseteq F|X_1$ and thus there is a free Borel action a' of $\mathbb{F}_\infty$ on X_1 with $E_{a'} \subseteq F|X_1$. Put $F' = E_{a'} \oplus F|X_0 \subseteq F$. Then $C(F') = C_{\mu|X_1}(E_{a'}) + C_{\mu|X_0}(F|X_0) = \infty$, since $C_{\mu|X_1}(E_{a'}) = \infty$. □

It is clear that (i) and (ii) are contradictory, so let us assume that (i) fails for E and then show (ii). By Lemma 12.3, we can assume that there is $F \in S(E)$ with $C(F) = \infty$. It follows that F is not smooth (see, e.g., [KM, Proposition 20.1]). Put $X_0 = \{x\colon |[x]_E| = \infty\}, X_1 = X \setminus X_0$. Thus $\mu(X_0) > 0$. Now $F|X_1$ is smooth and thus $C_{\mu|X_1}(F|X_1) < \infty$. Since

$$C(F) = C_{\mu|X_0}(F|X_0) + C_{\mu|X_1}(F|X_1) = \infty,$$

it follows that $C_{\mu|X_0}(F|X_0) = \infty$.

Fix now $\epsilon > 0$. Let $S \subseteq X_0$ be a complete section of $F|X_0$ such that $\mu(S) < \epsilon$. We have

$$C_{\mu|X_0}(F|X_0) = C_{\mu|S}(F|S) + \mu(X_0 \setminus S) = \infty,$$

so $C_{\mu|S}(F|S) = \infty$.

Let now $R \in S(E)$. Put $R_\epsilon = R|(X \setminus S) \oplus F|S$. Then

$$C(R_\epsilon) = C_{\mu|(X\setminus S)}(R|(X \setminus S)) + C_{\mu|S}(F|S) = \infty.$$

Also for any $T \in [E]$, we have

$$\begin{aligned} A_{T,R_\epsilon} =& \{x\colon x \notin S \,\&\, T(x) \notin S \,\&\, x \in A_{T,R}\} \\ & \sqcup \{(x\colon (x \in S \ \vee\ T(x) \in S) \,\&\, x \in A_{T,R_\epsilon}\}, \end{aligned}$$

$$\begin{aligned} A_{T,R} =& \{x\colon x \notin S \,\&\, T(x) \notin S \,\&\, x \in A_{T,R}\} \\ & \sqcup \{(x\colon (x \in S \ \vee\ T(x) \in S) \,\&\, x \in A_{T,R}\}, \end{aligned}$$

so $A_{T,R_\epsilon} \Delta A_{T,R} \subseteq S \cup T^{-1}(S)$ and therefore $\mu(A_{T,R_\epsilon} \Delta A_{T,R}) < 2\epsilon$. It follows that $R_{\frac{1}{n}}$ converges in the uniform topology to R. □

Remark 12.4. It is unknown if condition (i) in Theorem 12.2 is equivalent to hyperfiniteness.

The following problem is open. For convenience, we will say that E is of **type II** if it is aperiodic and there is $F \in S(E)$ with $C(F) > 1$.

Problem 12.5. *Let E be a type II equivalence relation. Is $InfCost_E$ comeager in $S(E)$?*

We will next consider the descriptive complexity of the cost function.

Proposition 12.6. *The set $FinCost_E$ is analytic in $S(E)$ and the cost function $F \mapsto C(F)$ is Borel on $FinCost_E$.*

Proof. The first assertion follows by a direct calculation (or using Proposition 19.1 and Proposition 19.11 below).

For the second assertion, we recall that if an ergodic $F \in S(E)$ has finite cost, then it is induced by an action of some $\mathbb{F}_n$ (see [KM, Lemma 27.7]). Also the cost function $a \in A(\mathbb{F}_n, E) \mapsto C(a) = C(E_a)$ is upper semicontinuous on $A(\mathbb{F}_n, E)$ by [K, First Remark in page 78]. Thus for ergodic $F \in S(E)$ of finite cost and $r \in \mathbb{R}$, we have:

$$\begin{aligned} C(F) < r &\iff \exists n \exists a \in A(\mathbb{F}_n, E)(E_a = F \;\&\; C(a) < r) \\ &\iff \forall n \forall a \in A(\mathbb{F}_n, E)(E_a = F \implies C(a) < r), \end{aligned}$$

which shows that the cost function is Borel on the set $\mathcal{ERG}_E \cap FinCost_E$. The general case can be proved using the Ergodic Decomposition Theorem 4.19, Theorem 4.20 and the integration formula for cost with respect to the ergodic decomposition [KM, Corollary 18.6], which, in particular, shows that if an equivalence relation has finite cost, so do (almost) all its ergodic components. $\square$

The following is an open problem:

Problem 12.7. *Is the cost function $F \mapsto C(F)$ Borel on $S(E)$? Equivalently is the set $FinCost_E$ Borel in $S(E)$?*

We next notice some related facts and questions. It is clear from Theorem 12.2 that for each E of type II the sets $\{F \in S(E) \colon C(F) > r\}, \{F \in S(E) \colon C(F) \geq r\}$, for $r \in \mathbb{R}, r > 0$, are not uniformly closed. We can also see that for some E the sets $\{F \in S(E) \colon C(F) < r\}, r > 1, \{F \in S(E) \colon C(F) \leq r\}, r \geq 1$, are not closed. Take $n > r$, let $\Gamma = \mathbb{F}_n \times \mathbb{Z}$, let $a \in \mathrm{FR}(\Gamma, X, \mu)$ and let $E_a \subseteq E$. Put $\Gamma_m = 2^m\mathbb{Z}, m \geq 1$, and let $E_m = E_{a|(\mathbb{F}_n \times \Gamma_m)}$, so that $E_1 \supseteq E_2 \dots$ and $C(E_m) = 1$. Now $\bigcap_m E_m = E_{a|\mathbb{F}_n}$, so $E_m \to E_{a|\mathbb{F}_n}$ and $C(E_{a|\mathbb{F}_n}) = n > r$. A similar argument, using $\Gamma = \mathbb{F}_\infty \times \mathbb{Z}$, shows that in general $\{F \in S(E) \colon C(F) < \infty\}$ is not closed. The following problem is open:

Problem 12.8. *Are the sets*

$$\{F \in S(E) \colon C(F) < r\}, r > 1,$$
$$\{F \in S(E) \colon C(F) < \infty\},$$
$$\{F \in S(E) \colon C(F) \leq r\}, r \geq 1.$$

uniformly closed?

One can also use these observations to answer a question that arises from [K, First Remark in page 78]. It is shown there that when the infinite group Γ is finitely generated, the cost function C on $A(\Gamma, E)$ is upper semicontinuous. Is that true for arbitrary infinite Γ? The answer is negative:

Proposition 12.9. *For any equivalence relation E of type II, the function $a \in A(\mathbb{F}_\infty, E) \mapsto C(a)$ is not upper semicontinuous.*

Proof. By Theorem 7.1, there is a continuous map $\Psi \colon S(E) \to A(\mathbb{F}_\infty, E)$ such that $E_{\Psi(F)} = F$. So if the cost function was upper semicontinuous in $A(\mathbb{F}_\infty, E)$, for each $r \in \mathbb{R}$ the set $\{F \in S(E) \colon C(F) \geq r\}$ would be closed in $S(E)$, a contradiction. $\square$

Finally we show that an analog of Theorem 7.1 fails for $\mathbb{F}_n, n \geq 2$. Below let $\mathcal{F}_{n,E} = \{F \in S(E) \colon \exists a \in A(\mathbb{F}_n, E)[E_a = E]\}$.

Proposition 12.10. *Let $n \geq 2$. If E is of type II, there is no continuous function $\Psi_n \colon \mathcal{F}_{n,E} \to A(\mathbb{F}_n, E)$ such that $E_{\Psi_n(F)} = F$.*

Proof. As in the proof of Lemma 12.3, there is an invariant Borel set X_1 of positive measure and a free Borel action a_∞ of $\mathbb{F}_\infty$ on X_1 with $E_{a_\infty} \subseteq E$. Let for $n \geq 1$, $a_n = a_\infty | \langle \gamma_0, \gamma_n \rangle$, where $\{\gamma_0, \gamma_1, \dots\}$ are free generators of $\mathbb{F}_\infty$. Let $X_0 = X \setminus X_1$ and put $R_0 = id|X_0$ and $F_n = E_{a_n} \oplus R_0$. Then $C(F_n) = 2\mu(X_1)$.

Lemma 12.11. *Let $a_0 = a_\infty | \langle \gamma_0 \rangle, F_0 = E_{a_0} \oplus R_0$. Then $F_n \to F_0$.*

Proof. Below put $\delta \cdot x = a_\infty(\delta, x)$. Fix $T \in [E]$. Then

$$A_{T,F_n} = \{x \in X_1 \colon T(x) \in \langle \gamma_0, \gamma_n \rangle \cdot x\} \sqcup \{x \in X_0 \colon T(x) = x\},$$

$$A_{T,F_0} = \{x \in X_1 \colon T(x) \in \langle \gamma_0 \rangle \cdot x\} \sqcup \{x \in X_0 \colon T(x) = x\}.$$

Thus $A_{T,F_0} \subseteq A_{T,F_n}$ and $(A_{T,F_n} \setminus A_{T,F_0}) \cap (A_{T,F_m} \setminus A_{T,F_0}) = \emptyset$, if $n \neq m$, so $\mu(A_{T,F_n} \setminus A_{T,F_0}) \to 0$, thus $\mu(A_{T,F_n}) \to \mu(A_{T,F_n})$. □

Note also that $C(F_0) = \mu(X_1)$. If such Ψ_n existed, and since the cost function is upper semicontinuous on $A(\mathbb{F}_n, E)$, the set $\{F \in S(E) \colon C(F) \geq r\}$ would be closed in $\mathcal{F}_{n,E}$. Taking $r = 2\mu(X_1)$ we have a contradiction. □

Notice that the set $\mathcal{F}_{n,E}$ is analytic in $S(E)$. The following problem is open:

Problem 12.12. *Let $n \geq 2$. Is there a Borel function $\Psi_n \colon \mathcal{F}_{n,E} \to A(\mathbb{F}_n, E)$ such that $E_{\Psi_n(F)} = F$?*

For $n = 1$, $\mathcal{F}_{1,E} = \mathcal{H}_E$, thus, by Theorem 8.1, $\mathcal{F}_{1,E}$ is closed in $S(E)$ and we will see in Theorem 14.1 that Problem 12.12 has a positive solution for $n = 1$. (Note that $A(\mathbb{F}_1, E) = A(\mathbb{Z}, E)$ is homeomorphic to $[E]$.) However we do not know if there is continuous $\Psi_1 \colon \mathcal{F}_{1,E} \to A(\mathbb{Z}, E)$ with $E_{\Psi_1(F)} = F$.

13. Normality

We discuss here normal subequivalence relations, see [FSZ]. Let E be ergodic and let $N = [E : F] \leq \infty$ be the **index** of F in E, i.e., the number of F-classes in each E-class. A sequence $(\varphi_n)_{n<N}$ of Borel functions on X such that for each x, $([\varphi_n(x)]_F)_{n<N}$ is an injective enumeration of the F-classes in $[x]_E$ is called a **choice sequence**. Again we identify two such sequences if they agree a.e. Every F admits a choice sequence and if F is also ergodic, then such $(\varphi_n)_{n<N}$ can be found which are in $\mathrm{Aut}(X, \mu)$ (see [FSZ, Lemma 1.3]).

Definition 13.1. *Let E be ergodic. A subequivalence relation $F \in S(E)$ is* **normal** *in E, in symbols*

$$F \lhd E,$$

if there are choice sequences which are F-invariant.

In particular, if $F \lhd E$ and F is ergodic, then one can find choice sequences which are F-invariant and in $\mathrm{Aut}(X, \mu)$. We now have the following result concerning the complexity of the set of normal subequivalence relations.

Theorem 13.2. *The set $Normal(E)$ of normal subequivalence relations of an ergodic equivalence relation E is Borel in $S(E)$.*

Proof. We first note the following fact:

Lemma 13.3. *The set $\{F \in S(E) \colon [E : F] = N\}$ is $F_{\sigma\delta}$, for any $N \leq \infty$.*

Proof. For $N = \infty$ this follows from Proposition 9.1. So we can assume that $N < \infty$. Then the proof is similar to that of Proposition 9.4. Let (T_i) be a generating sequence for E. Then we have that $[F : E] \leq N$ iff

$$\forall k \exists M \forall n \left(\mu(\{x \colon \exists s \in M^N \forall j \leq n \exists k < N (T_j(x) F T_{s(k)}(x))\}) \geq 1 - \frac{1}{k} \right).$$

□

So it is enough to show that for each $N \leq \infty$, the set $\{F \in S(E) \colon [E : F] = N \ \& \ F \lhd E\}$ is Borel.

We will first deal with ergodic normal subequivalence relations and then consider the general case.

Ergodic case. The set $\{F \in \mathcal{ERG}_E \colon [E : F] = N \ \& \ F \lhd E\}$ *is Borel in* $S(E)$.

We will view below E as a genuine countable Borel equivalence relation (and not one defined a.e.). Let then $R \subseteq S(E) \times E$ be as in Proposition 4.18, so that for each $F \in S(E)$, $R_F = F^0$ is a subequivalence relation of E which is a representative for F in $S(E)$. Let $\Gamma = \{\gamma_n\}$ be a countable group acting in a Borel way on X generating E. Then define inductively for each $F \in S(E), n < N$, a Borel function $\varphi_n^F \colon X \to X$ as follows:

$$\varphi_0^F(x) = x$$

$$\varphi_n^F(x) = \gamma_k \cdot x,$$

where k is least such that $\gamma_k \cdot x \notin [\varphi_i^F(x)]_{F^0}, \forall i < n$, if such exists; else $\varphi_n^F(x) = x$. Clearly $(\varphi_n^F)_{n<N}$ is a choice sequence for F (a.e.). Moreover the relation $Q \subseteq S(E) \times \mathbb{N} \times X^2$, given by:

$$Q(F, n, x, y) \iff \varphi_n^F(x) = y$$

is Borel.

Define now for each $F \in S(E)$, a function $\sigma_F : E \to S_N$, where S_N is the symmetric group on N elements, as follows:

$$\sigma_F(x, y)(i) = j \iff \varphi_i^F(x) F^0 \varphi_j^F(y),$$

provided that there are exactly N F^0-classes in $[x]_E = [y]_E$; else $\sigma_F(x, y)(i) = i$. Then σ_F is a Borel cocycle from E into S_N and is the **index cocycle** of F corresponding to the choice sequence $(\varphi_n^F)_{n<N}$ (a.e.) (see [FSZ, Lemma 1.2]).

From [FSZ, Definition 2.1 and Theorem 2.2], we have that $F \lhd E$ iff $\sigma_F|F$ is a **coboundary**, i.e., there is a function $f \in L(X, \mu, S_N)$ such that for xFy, $\sigma_F(x, y) = f(y)f(x)^{-1}$ (a.e.). Here $L(X, \mu, S_N)$ is the space of Borel functions from X to the Polish group S_N, two functions being identified if they agree a.e. Then $L(X, \mu, S_N)$ is a Polish group under pointwise multiplication and the topology of convergence in measure. Let also

$Z^1(F, S_N)$ be the Polish space of Borel cocycles from F to S_N (two such cocycles being identified if they agree a.e.), see [K, Section 24]. The Polish group $L(X, \mu, S_N)$ acts continuously on $Z^1(F, S_N)$ via $f \cdot \alpha(x, y) = f(y)\alpha(x, y)f(x)^{-1}$. Denoting by 1 the trivial cocycle (that sends any $(x, y) \in F$ to the identity element 1 of S_N), we thus have that α is a coboundary iff it is in the orbit of 1 in the action of $L(X, \mu, S_N)$.

The stabilizer of 1 in this action consists of all $f \in L(X, \mu, S_N)$, which are F-invariant and thus constant, if F is ergodic. Thus for ergodic F this stabilizer is equal to the group S_N (identified with the group of constant functions from X to S_N). Clearly S_N is a closed subgroup of $L(X, \mu, S_N)$, so let T be a Borel set that contains exactly one element in each left-coset of S_N in $L(X, \mu, S_N)$. Then if α is a coboundary there is a unique $f \in T$ such that $f \cdot 1 = \alpha$. Define then $P \subseteq \{F \in \mathcal{ERG}_E \colon [E : F] = N\} \times L(X, \mu, S_N)$ by

$$P(F, f) \iff f \in T \,\&\, f \cdot 1 = \sigma_F | F.$$

Then by the preceding discussion the first projection map is an injective map from P onto $\{F \in \mathcal{ERG}_E \colon [E : F] = N \,\&\, F \lhd E\}$. It thus suffices to show that P is a Borel set or that

$$S(F, f) \iff f \cdot 1 = \sigma_F | F.$$

is Borel.

Recall that $L(X, \mu, S_N)$ admits the compatible complete metric

$$d(f, g) = \int D(f(x), g(x)) d\mu(x),$$

where D is the usual compatible metric for S_N (which is bounded by 1).

Lemma 13.4. *For $f \in L(X, \mu, S_N)$, let for $m, n < N$,*

$$A_{f,m,n} = \{x \colon f(x)(m) = n\} \in \mathrm{MALG}_\mu.$$

Then $f \in L(X, \mu, S_N) \to A_{f,m,n} \in \mathrm{MALG}_\mu$ is Lipschitz (for the usual metric ρ on MALG_μ).

Proof. Let ϵ be such that

$$D(p, q) < \epsilon \implies p(m) = q(m).$$

Then we will show that

$$\rho(A_{f,m,n}, A_{g,m,n}) \leq \frac{d(f,g)}{\epsilon}.$$

Let $d(f,g) = a$. Then by Markov's inequality

$$\mu(\{x\colon D(f(x), g(x)) \geq \epsilon\})) \leq \frac{a}{\epsilon}.$$

Now

$$A_{f,m,n} \Delta A_{g,m,n} \subseteq \{x\colon D(f(x), g(x)) \geq \epsilon\},$$

so $\rho(A_{f,m,n}, A_{g,m,n}) \leq \frac{a}{\epsilon}$. □

Lemma 13.5. *There is a Borel set $U \subseteq L(X,\mu,S_N) \times X \times N^2$ such that for each $f \in L(X,\mu,S_N), x \in X$, the section $U_{f,x}$ is the graph of a permutation $p_{f,x} \in S_N$ and the map $f^0\colon X \to S_N$ given by $f^0(x) = p_{f,x}$ is equal to f a.e.*

Proof. We can assume that $X = [0,1]$ and μ is Lebesgue measure. Let $A^*_{f,m,n}$ be the set of density points of $A_{f,m,n}$. Then by Lemma 13.4, the relation $U^*(f,x,m,n) \iff x \in A^*_{f,m,n}$ is Borel. Finally let

$$\begin{aligned} U(f,x,m,n) \iff & (U^*_{f,x} \text{ is not the graph of an element of } S_N \text{ and } m = n) \\ & \text{or (it is such a graph and } U^*(f,x,m,n)). \end{aligned}$$

□

We have now that

$$S(F,f) \iff \forall i \forall^* x[xFT_i(x) \implies \sigma_F(x, T_i(x)) = f(T_i(x))f(x)^{-1}],$$

where $\forall^* x$ means "for almost all x." So $S(F,f)$ is equivalent to

$$\forall i \forall m \forall^* x[xFT_i(x)) \implies \varphi^F_m(x) F \varphi_{f(T_i(x))f(x)^{-1}(m)}(T_i(x))]$$

and therefore to

$$\forall i \forall m \forall^* x \exists j,k (xFT_i(x) \implies$$
$$[\{\varphi^F_m(x) = T_j(x) \mathbin{\&} \varphi_{f^0(T_i(x))f^0(x)^{-1}(m)}(T_i(x)) = T_k(x)\} \mathbin{\&} T_j(x)FT_k(x)]).$$

Let B be the Borel set of x satisfying the condition within $\{\dots\}$ in the line above, so that finally

$$S(F,f) \iff \forall i \forall m \forall^* x \exists j,k[x \notin A_{T_i,F} \text{ or } (x \in B \mathbin{\&} x \in A_{T_j,T_k,F})],$$

and so $S(F, f)$ is equivalent to:

$$\forall i \forall m \forall n \exists M \mu(\{x\colon \exists j, k \leq M[x \notin A_{T_i,F} \text{ or } (x \in B \;\&\; x \in A_{T_j,T_k,F})]\}) \geq 1 - \frac{1}{n}.$$

Since the maps $F \mapsto A_{T_i,F}, A_{T_j,T_k,F}$ from $S(E)$ to MALG_μ are continuous, this shows that S is Borel and completes the proof in the ergodic case.

General case. The set $\{F \in S(E)\colon [F : E] = N \;\&\; F \lhd E\}$ *is Borel in* $S(E)$.

Repeating the argument as in the ergodic case, we note that the stabilizer of 1 is the closed subgroup G_F of the F-invariant functions in the space $L(X, \mu, S_N)$. Again as in the previous argument, it is enough to find a Borel transversal T_F for the cosets of G_F in $L(X, \mu, S_N)$, so that relation

$$T(F, f) \iff f \in T_F$$

is Borel (as a subset of $S(E) \times L(X, \mu, S_N)$). Denote by $\mathcal{F}$ the Effros Borel space of the closed subgroups of $L(X, \mu, S_N)$. By the usual proof of the existence of a Borel transversal for the cosets of a closed subgroup of a Polish group, it is then enough to show that the map $F \in S(E) \mapsto G_F \in \mathcal{F}$ is Borel or equivalently that there is a Borel function

$$\delta\colon S(E) \to L(X, \mu, S_N)^{\mathbb{N}}$$

such that for each $F \in S(E)$ the sequence $\delta(F)$ is dense in G_F.

To see this consider the Ergodic Decomposition Theorem 4.19 and Theorem 4.20, whose notation we use below. Thus π_F is an ergodic decomposition of F^0, mapping X to $P(X)$, and has range the set $\mathcal{EI}_{F^0}$.

Then $f \in L(X, \mu, S_N)$ is F-invariant iff it is of the form $g \circ \pi_F$ for a uniquely determined $g \in L(P(X), (\pi_F)_*(\mu), S_N)$. Thus the map $g \in L(P(X), (\pi_F)_*(\mu), S_N) \mapsto g \circ \pi_F \in L(X, \mu, S_N)$ is an isometric embedding, whose range is G_F.

Now pick a countable Boolean algebra $\mathcal{B}$ of Borel subsets of $P(X)$ which generates its Borel sets. Then for any probability Borel measure ν on $P(X)$, $\mathcal{B}$ is dense in the measure algebra MALG_ν. Fix also a countable dense set $\Sigma = \{\sigma_n\}$ in S_N. Then the Borel maps from $P(X)$ into S_N that are constant in the pieces of a partition of $P(X)$ in $\mathcal{B}$ and take values in Σ form a dense set in any $L(P(X), (\pi_F)_*(\mu), S_N)$. Enumerate these functions as $\{g_0, g_1, \dots\}$.

Finally define the function $\delta = (\delta_n)$ as follows:

$$\delta_n(F) = g_n \circ \pi_F.$$

It only remains to check that this is a Borel function and for that we verify that for any n, any (genuine) Borel function h_0 from X to S_N and any $\epsilon > 0$, the set of all $F \in S(E)$ for which

$$d(\delta_n(F), h_0) = \int D(\delta_n(F)(x), h_0(x))dx < \epsilon$$

is Borel, which is clear as the function $(F, x) \mapsto D(g_n(\pi_F(x)), h_0(x))$ is Borel.

$\square$

14. A selection theorem for hyperfiniteness

Recall that $\mathcal{H}$ is the class of hyperfinite equivalence relations. For each E, the set $\mathcal{H}_E$ is closed in $S(E)$ by Theorem 8.1. Also the set $\mathcal{ERGH}_E$ of ergodic hyperfinite subequivalence relations of E is a G_δ set in $S(E)$ by Theorem 8.1 and Theorem 10.1. Note that if F is in $\mathcal{ERG}$, then F is aperiodic.

We next prove the following selection result.

Theorem 14.1. *There is a Borel function $\Theta\colon \mathcal{H}_E \to [E]$ such that for $F \in \mathcal{H}_E$, if $\Theta(F) = T$, then $F = E_T$ (i.e., $xFy \iff \exists n \in \mathbb{Z}(T^n(x) = y)$).*

Proof. We will first give a detailed argument that there is a Borel function $\Phi\colon \mathcal{ERGH}_E \to [E]$ such that for $F \in \mathcal{ERGH}_E$, if $\Phi(F) = T$, then $F = E_T$, i.e, we will first prove the theorem for the ergodic hyperfinite equivalence relations. Then we will indicate how this can be extended to all hyperfinite equivalence relations.

Let for $F \in S(E)$,

$$A(F) = \{T \in [F]\colon T \text{ is aperiodic}\}.$$

Then for any aperiodic F, $A(F)$ is a closed non-empty subset of $[F]$ (and thus of $[E]$); see [K, 3.5]. We first prove the following:

Lemma 14.2. *The following are equivalent:*

(i) There is a Borel function $\Phi\colon \mathcal{ERGH}_E \to [E]$ such that if $\Phi(F) = T$, then $E_T = F$ (thus $T \in A(F)$).

(ii) The function $A|\mathcal{ERGH}_E$ from $\mathcal{ERGH}_E$ to $\mathcal{F}^*([E])$ is Borel.

(iii) There is a Borel function $\Omega\colon \mathcal{ERGH}_E \to [E]$ such that $\Omega(F) \in A(F)$.

Proof. (ii) $\Rightarrow$ (i). We need the following:

Sublemma 14.3. *Let $F \in \mathcal{ERGH}_E$. Then the generic element $T \in A(F)$ has the property that $E_T = F$.*

Proof. Let $C = \{T \in A(F) \colon E_T = F\}$. We show first that it is dense in $A(F)$. To see this, fix $T_0 \in A(F)$ with $E_{T_0} = F$. Then the orbit of T_0 under the conjugation action of $[F]$ on $A(F)$ is dense in $A(F)$, by [K], 3.4. Clearly every element T of that orbit has $E_T = F$.

It remains to show that C is G_δ in $A(F)$. For that it is enough to show that the map $[E] \ni T \mapsto E_T \in S(E)$ is of Baire class 1. This will follow if we can show that for any $S \in [E]$ and $\alpha \in \mathbb{R}$, the set $\{T \in [E] \colon \alpha < \mu(A_{S,E_T})\}$ is open.

Now

$$\begin{aligned} A_{S,E_T} &= \{x \colon (x, S(x)) \in E_T\} \\ &= \{x \colon \exists n \in \mathbb{Z}(S(x) = T^n(x)) \\ &= \bigcup_{N \in \mathbb{N}} \{x \colon \exists |n| \leq N(S(x) = T^n(x))\} \\ &= \bigcup_{N \in \mathbb{N}} A_N^T, \end{aligned}$$

where $A_N^T = \{x \colon \exists |n| \leq N(S(x) = T^n(x))\}$. Clearly $A_0^T \subseteq A_1^T \subseteq \dots$, so

$$\alpha < \mu(A_{S,E_T}) \iff \exists N(\mu(A_N^T) > \alpha),$$

thus it suffices to show that

$$\{T \colon \mu(A_N^T) > \alpha\}$$

is open in $[E]$. Fix T_1 such that $\mu(A_N^{T_1}) > \alpha$ and let $\delta = \mu(A_N^{T_1}) - \alpha > 0$. Then let $\epsilon > 0$ be such that $N(N+1)\epsilon < \delta$. We will show that if $d_u(T, T_1) < \epsilon$, then $\mu(A_N^T) > \alpha$.

If $d_u(T, T_1) < \epsilon$, then $d_u(T^n, T_1^n) < |n|\epsilon$, for any $n \in \mathbb{Z}$. Since

$$\{x \colon S(x) = T^n(x)\} \Delta \{x \colon S(x) = T_1^n(x)\} \subseteq \{x \colon T^n(x) \neq T_1^n(x)\},$$

we have

$$\mu(\{x \colon S(x) = T^n(x)\} \Delta \{x \colon S(x) = T_1^n(x)\}) < |n|\epsilon,$$

so

$$\mu(\{x\colon \exists |n| \leq N(S(x) = T^n(x))\}\Delta\{x\colon \exists |n| \leq N(S(x) = T_1^n(x))\})$$
$$\leq \sum_{|n|\leq N}(|n|\epsilon) = N(N+1)\epsilon < \delta,$$

therefore
$$\mu(A_N^T) > \mu(A_N^{T_1}) - \delta = \alpha.$$

This concludes the proof of the Sublemma. □

Consider now the relation $P \subseteq \mathcal{ERGH}_E \times [E]$ given by

$$P(F,T) \iff T \in A(F) \;\&\; E_T = F.$$

Clearly it is Borel and our goal is to find a Borel uniformizing function Φ for P. To each $F \in \mathcal{ERGH}_E$ assign the σ-ideal $\mathcal{I}_F$ on $[E]$ defined by

$$\mathcal{I}_F = \{W \subseteq [E]\colon W \cap A(F) \text{ is meager in } A(F)\}.$$

It is clear that for $F \in \mathcal{ERGH}_F$, $P_F = \{T\colon P(F,T)\} \notin \mathcal{I}_F$. Therefore by [K2, 18.6], it is enough to show that $F \mapsto \mathcal{I}_F$ is Borel on Borel. So let Z be a Polish space and $U \subseteq Z \times \mathcal{ERGH}_F \times [E]$ be Borel in order to show that

$$\{(z,F)\colon U_{z,F} \text{ is meager in } A(F)\}$$

is Borel. In fact, more generally, we will show that for any $W \subseteq [E]$, which is an open non-empty set in $[E]$, the set

$$M_{U,W} = \{(z,F)\colon A(F) \cap W \neq \emptyset \;\&\; U_{z,F} \text{ is not meager in } A(F) \cap W\}.$$

is Borel. Note that if $\{W_n\}$ is a basis of nonempty open sets in $[E]$, then we have for Borel $U, U_n \subseteq Z \times \mathcal{ERGH}_F \times [E]$:

$$M_{\bigcup_n U_n, W} = \bigcup_n M_{U_n,W}$$

and (letting $\sim U = (Z \times \mathcal{ERGH}_E \times [E]) \setminus U$)

$$M_{\sim U,W} = [(Z \times \mathcal{ERGH}_E) \setminus \bigcap\{M_{U,W_n}\colon W_n \subseteq W, W_n \cap A(F) \neq \emptyset\}]$$
$$\cap \{(z,F) \in Z \times \mathcal{ERGH}_F\colon A(F) \cap W \neq \emptyset\},$$

thus, since $\{(z, F) \in Z \times \mathcal{ERGH}_E \colon A(F) \cap W \neq \emptyset\}$ is Borel by our hypothesis, it is enough to show that $M_{U,W}$ is Borel for each $U = U_1 \times U_2 \times U_3$, where U_1 is open in Z, U_2 is open in $\mathcal{ERGH}_E$ and U_3 is open in $[E]$. But in that case

$$\begin{aligned}(z, F) \in M_{U,W} \iff & z \in U_1 \;\&\; F \in U_2 \;\&\; A(F) \cap W \neq \emptyset \;\&\\ & U_3 \text{ is not meager in } A(F) \cap W\\ \iff & z \in U_1 \;\&\; F \in U_2 \;\&\; A(F) \cap W \neq \emptyset \;\&\\ & A(F) \cap W \cap U_3 \neq \emptyset,\end{aligned}$$

which again is Borel by hypothesis.

(i) $\Rightarrow$ (iii): Obvious taking $\Omega = \Phi$.

(iii) $\Rightarrow$ (ii): By [K, 3.4], the conjugacy class $\{T\,\Omega(F)\,T^{-1} \colon T \in [F]\}$ is dense in $A(F)$. So for $W \subseteq [E]$ open,

$$\begin{aligned}A(F) \cap W \neq \emptyset \iff & \exists T \in [F](T\,\Omega(F)\,T^{-1} \in W)\\ \iff & \exists T \in \mathcal{D}(T\,\Omega(F)\,T^{-1} \in W),\end{aligned}$$

for any countable dense subset $\mathcal{D} \subseteq [F]$. It is thus enough to show that there is a Borel function $D \colon S(E) \to [E]^{\mathbb{N}}$ such that $D(F) = (T_n)_{n\in\mathbb{N}}$, where $\{T_n\}_{n\in\mathbb{N}}$ is dense in $[F]$. Since $F \in S(E)$ is identified with $[F]$, a closed subset of $[E]$, this follows from [K2, 12.13].

This concludes the proof of the Lemma. □

Thus to complete the proof of Theorem 14.1 in the ergodic case, it is enough to prove (iii) of the preceding lemma.

We now use Proposition 4.18, in which we recall that E is viewed as a genuine equivalence relation and not one viewed a.e., Combining this with the proof of [K, 3.5], we then have:

Lemma 14.4. *There is a Borel set $Q \subseteq \mathcal{ERGH}_E \times E$ such that for any $F \in \mathcal{ERGH}_E, Q_F \subseteq F^\circ$ and Q_F is the graph of a Borel automorphism T_F of X (thus $E_{T_F} \subseteq F^\circ$) such that T_F restricted to the aperiodic part of F° (i.e., the set of all x with $[x]_{F^\circ}$ infinite) is also aperiodic.*

In particular, if $\langle T_F \rangle$ is the element of $[F^\circ] = [F]$ represented by T_F, then $\langle T_F \rangle \in A(F)$. We put $\Omega(F) = \langle T_F \rangle$ for $F \in \mathcal{ERGH}_E$. It remains to verify that $\Omega \colon \mathcal{ERGH}_E \to [E]$ is Borel.

Fix $T_0 \in [E]$. It is enough to show that

$$\{F \in \mathcal{ERGH}_E \colon d(T_F, T_0) < \epsilon\}$$

is Borel in $S(E)$. Now for $F \in \mathcal{ERGH}_E$,

$$\begin{aligned} d_u(T_F, T_0) < \epsilon &\iff \mu(\{x \colon T_F(x) \neq T_0(x)\}) < \epsilon \\ &\iff \mu(\{x \colon (x, T_0(x)) \notin Q_F\}) < \epsilon, \\ &\iff \mu(\{x \colon (F, x, T_0(x)) \notin Q\}) < \epsilon, \end{aligned}$$

which is clearly a Borel condition on F.

This completes the proof of selection for the ergodic case.

The proof in the general case can proceed in two different ways. The first is by using the ergodic composition theorem, see Theorem 4.20. The second uses a result of Miri Segal in her (unpublished) Ph.D. Thesis (see [K4, 8.47 and the following paragraph]. I would like to thank Ben Miller for this suggestion. Segal's result states that for each (genuine) countable Borel equivalence relation F, which is hyperfinite μ-a.e., one can find in an effective Borel way a Borel automorphism that generates F μ-a.e. Combined with Proposition 4.18 this implies the following:

Proposition 14.5. *There is a Borel set $P \subseteq \mathcal{H}_E \times E$ such that for any $F \in \mathcal{H}_E, P_F \subseteq F^\circ$ and P_F is the graph of a Borel automorphism T_F of X such that E_{T_F} is equal to F in $S(E)$.*

This together with the argument following Lemma 14.4 completes the proof of Theorem 14.1. $\square$

Combining Proposition 8.3 with Theorem 14.1 and the proof of [DJK, Theorem 5.1], we also have the following result:

Theorem 14.6. *There is a Borel function $H \colon \mathcal{H}_E \to S(E)^{\mathbb{N}}$ such that for $F \in \mathcal{H}_E$ we have that for each n, $H(F)_n \in \mathcal{BF}_E$, $H(F)_n \subseteq H(F)_{n+1}$, and $F = \bigcup_n H(F)_n$.*

15. Invariant, random equivalence relations on groups

We study here the connection between the space of subequivalence relations and that of invariant, random equivalence relations on groups.

15.1 Equivalence relations on groups

For each infinite countable group Γ, denote by $\mathrm{Eq}(\Gamma)$ the space of equivalence relations on Γ. This is a compact subspace of 2^{Γ^2}. The group Γ acts continuously by translation on $\mathrm{Eq}(\Gamma)$: if $\gamma \in \Gamma, e \in \mathrm{Eq}(\Gamma)$, then

$$(\delta, \epsilon) \in \gamma \cdot e \iff (\gamma^{-1}\delta, \gamma^{-1}\epsilon) \in e.$$

Let σ be a Borel probability measure on $\mathrm{Eq}(\Gamma)$. If σ is invariant under the action of Γ, we say that σ is a (Γ-)**invariant, random equivalence relation (IRE)** on Γ. We denote by $\mathrm{IRE}(\Gamma)$ the space of these measures.

Clearly $\mathrm{IRE}(\Gamma)$ is a compact subspace of the space of all Borel probability measures on $\mathrm{Eq}(\Gamma)$ (which is equipped, as usual, with the weak*-topology, in which it is compact metrizable)

There is a canonical connection between subequivalence relations of the equivalence relation E_a induced by an action $a \in A(\Gamma, X, \mu)$ and IRE on Γ, which is a special case of structurability of such equivalence relations. See [KM, 29.1], [CK, Section 2], and [T-D, Appendix A] for the particular case of equivalence relations.

Let $a \in A(\Gamma, X, \mu)$ and put $E = E_a$. Given $F \in S(E)$, define the map

$$e_F^a = e_F \colon X \to \mathrm{Eq}(\Gamma)$$

by

$$(\gamma, \delta) \in e_F(x) \iff (\gamma^{-1} \cdot x, \delta^{-1} \cdot x) \in F.$$

Then e_F is a Γ-equivariant Borel function. Put

$$\sigma^a(F) = \sigma(F) = (e_F)_*\mu.$$

Thus $\sigma^a(F) \in \mathrm{IRE}(\Gamma)$.

Proposition 15.1. *The map* $\sigma^a \colon S(E) \to \mathrm{IRE}(\Gamma)$ *is continuous.*

Proof. Fix $\alpha_i, \beta_i, \gamma_j, \delta_j \in \Gamma$, $i \leq m$, $j \leq k$, and put

$$A^a_{\bar\alpha,\bar\beta,\bar\gamma,\bar\delta,F} = A_{\bar\alpha,\bar\beta,\bar\gamma,\bar\delta,F} = \bigcap_{i\leq m} A_{(\alpha_i^a)^{-1},(\beta_i^a)^{-1},F} \cap \bigcap_{j\leq k}(X \setminus A_{(\gamma_j^a)^{-1},(\delta_j^a)^{-1},F}),$$

where $A_{S,T,F}$, for $S, T \in [E]$, is defined in the proof of Proposition 4.29.

It is enough to prove that the map that sends $F \in S(E)$ to the real number

$$\sigma^a(F)(\{e \in \mathrm{Eq}(\Gamma) \colon \forall i \leq m(\alpha_i, \beta_i) \in e \;\&\; \forall j \leq k(\gamma_j, \delta_j) \notin e)\})$$

is continuous. But this number is equal to $\mu(A_{\bar\alpha,\bar\beta,\bar\gamma,\bar\delta,F})$, which depends continuously on F, since, by Proposition 4.29, the map $F \mapsto A_{S,T,F}$ as above is continuous. $\square$

Remark 15.2. The map σ^a is not injective. Consider, for example, the shift action s of Γ on $[0,1]^\Gamma$, with the usual product measure. Let $F_1 = E_s \cap \{(x,y) \colon x(1) = y(1)\}$, $F_2 = E_s \cap \{(x,y) \colon x(\gamma) = y(\gamma)\}$, where $\gamma \neq 1$. Then $e_{F_1} = e_{F_2}$ is the constant function with value the equality relation $=_\Gamma$ on Γ, so $\sigma(F_1) = \sigma(F_2)$ is the Dirac measure at $=_\Gamma$ but $F_1 \neq F_2$.

It turns out that every IRE is generated by the above procedure for some, in fact free, action a and equivalence relation F. Below we denote by $\mathrm{FR}(\Gamma, X, \mu)$ the set of **free** actions in $A(\Gamma, X, \mu)$.

Proposition 15.3. $\mathrm{IRE}(\Gamma) = \{\sigma^a(F) \colon a \in A(\Gamma, X, \mu), F \in S(E_a)\}$
$= \{\sigma^a(F) \colon a \in \mathrm{FR}(\Gamma, X, \mu), F \in S(E_a)\}$.

Proof. Let $\sigma \in \mathrm{IRE}(\Gamma)$. Let $b \in \mathrm{FR}(\Gamma, Y, \nu)$ and put $X = \mathrm{Eq}(\Gamma) \times Y$, $\mu = \sigma \times \nu$. Let also a be the product action of Γ on X, so that $a \in \mathrm{FR}(\Gamma, X, \mu)$. Define $F \subseteq E_a$ by

$$(e, x) F (f, y) \iff \exists\gamma(\gamma \cdot (e, x) = (f, y) \;\&\; (1, \gamma^{-1}) \in e).$$

Then $e^a_F(e, x) = e$ and so $\sigma^a(F) = \sigma$. $\square$

A special case of the above construction of IRE is the following. Let Y be a standard Borel space and F a Borel equivalence relation on Y. Consider the product space $X = Y^\Gamma$ with the shift action s_Y of Γ on this space and let μ be a shift-invariant probability measure on X. Define the equivalence relation $\tilde{F}$ on X by $x\tilde{F}y \iff xE_{s_Y}y \ \&\ x(1)Fy(1)$. Let $e_{\tilde{F}}\colon X \to \mathrm{Eq}(\Gamma)$ be the associated map, so that $(\gamma, \delta) \in e_{\tilde{F}}(x) \iff x(\gamma)Fx(\delta)$. Finally consider the IRE $\sigma^{s_X}(\tilde{F})$.

Problem 15.4. *Is every element of* $\mathrm{IRE}(\Gamma)$ *of the form* $\sigma^{s_Y}(\tilde{F})$*, for some measure* μ *and Borel equivalence relation* F *on* Y*? What if we take* F *to be the equality relation on* Y*?*

Another way to obtain IRE is the following. Let $\mathrm{Sg}(\Gamma)$ be the space of subgroups of Γ, which is a compact subspace of 2^Γ on which Γ acts continuously by conjugation. An **invariant, random subgroup (IRS)** of Γ is a conjugation invariant Borel probability measure on $\mathrm{Sg}(\Gamma)$. Denote the space of such measures by $\mathrm{IRS}(\Gamma)$. There is a canonical homeomorphism Σ from $\mathrm{Sg}(\Gamma)$ into $\mathrm{Eq}(\Gamma)$ given by $(\gamma, \delta) \in \Sigma(H) \iff \gamma\delta^{-1} \in H$. Thus the equivalence classes of $\Sigma(H)$ are the right cosets of H. The range of Σ consists of the equivalence relations induced by the cosets of a subgroup of Γ. The embedding Σ is also Γ-equivariant, thus if $\mu \in \mathrm{IRS}(\Gamma)$, then $\Sigma_*\mu \in \mathrm{IRE}(\Gamma)$ and the range of Σ_* consists of the IRE that concentrate on the range of Σ. This forms a proper compact subset of $\mathrm{IRE}(\Gamma)$. Tucker-Drob [T-D, Appendix A] characterizes $\Sigma_*(\mathrm{IRS}(\Gamma))$ as consisting of exactly those $\sigma^a(F)$ for $F \subseteq E_a$ that are **normalized** by a, which means that each γ^a is an automorphism of F, i.e., $xFy \iff \gamma^a(x)F\gamma^a(y)$.

15.2 Classes of invariant, random equivalence relations

We say that $\sigma \in \mathrm{RS}(\Gamma)$ is an **aperiodic IRE** if it concentrates on the equivalence relations all of whose classes are infinite. It is an **infinite index IRE** if it concentrates on the equivalence relations that have infinitely many classes. Both the aperiodic and the infinite index IRE form G_δ sets in $\mathrm{IRE}(\Gamma)$. Similarly σ is a **finite index IRE** if it concentrates on the equivalence relations that have only finitely many classes. Finally, σ is a **finite IRE** if it concentrates on the equivalence relations all of whose classes are finite.

We now have the following results:

Theorem 15.5. *Let Γ be an infinite countable group. The generic* IRE *on Γ is aperiodic and has infinite index.*

Proof. By Proposition 15.1 and Proposition 15.3 and Theorem 8.2 the aperiodic IRE are dense in $\mathrm{IRE}(\Gamma)$ and by Proposition 9.1 the same is true for the infinite index IRE. □

Theorem 15.6. *Let Γ be an infinite amenable countable group. Then the finite index* IRE *are dense in* $\mathrm{IRE}(\Gamma)$.

Proof. This follows as before from Proposition 9.4. □

We do not know if this holds for all infinite Γ.

Theorem 15.7. *Let Γ be an infinite countable group. Then the following are equivalent:*

(i) Γ is amenable,

(ii) The finite IRE *are dense in* $\mathrm{IRE}(\Gamma)$,

(iii) The Dirac measure $\delta_{\Gamma\times\Gamma}$ on the equivalence relation $\Gamma \times \Gamma$ is a limit of finite IRE.

Proof. (i) $\implies$ (ii) follows from Proposition 15.1 and Proposition 15.3 and the paragraph following Theorem 8.1, while (ii) $\implies$ (iii) is obvious.

(iii) $\implies$ (i): Let σ_n be finite IRE such that $\sigma_n \to \delta_{\Gamma\times\Gamma}$. We will use these to find a left-invariant probability measure on Γ.

For $A \subseteq \Gamma$ and an equivalence relation e with finite classes, put

$$\rho_e(A) = \frac{|A \cap [1]_e|}{|[1]_e|}.$$

Then, for each n, put

$$\rho_n(A) = \int \rho_e(A) d\sigma_n(e).$$

Clearly ρ_n is a finitely additive probability measure on Γ.

Let now $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and put

$$\rho(A) = \lim_{n\to\mathcal{U}} \rho_n(A).$$

Again ρ is a finitely additive probability measure on Γ. We will show that it is left-invariant. We have for each $A \subseteq \Gamma, \gamma \in \Gamma$,

$$\rho(A) = \lim_{n \to \mathcal{U}} \int \frac{|A \cap [1]_e|}{|[1]_e|} d\sigma_n(e),$$

and

$$\rho(\gamma A) = \lim_{n \to \mathcal{U}} \int \frac{|\gamma A \cap [1]_e|}{|[1]_e|} d\sigma_n(e)$$

Now note that

$$\frac{|\gamma A \cap [1]_e|}{|[1]_e|} = \frac{|A \cap [\gamma^{-1}]_{\gamma^{-1} \cdot e}|}{|[\gamma^{-1}]_{\gamma^{-1} \cdot e}|},$$

so, using the invariance of ν_n, we have

$$\rho(\gamma A) = \lim_{n \to \mathcal{U}} \int \frac{|A \cap [\gamma^{-1}]_e|}{|[\gamma^{-1}]_e|} d\sigma_n(e).$$

It is thus enough to show that

$$\lim_{n \to \infty} \int \left(\frac{|A \cap [1]_e|}{|[1]_e|} - \frac{|A \cap [\gamma^{-1}]_e|}{|[\gamma^{-1}]_e|} \right) d\sigma_n(e) = 0.$$

Since $\sigma_n \to \delta_{\Gamma \times \Gamma}$, we have

$$\sigma_n(\{e \colon (1, \gamma^{-1}) \in e\}) \to \delta_{\Gamma \times \Gamma}(\{e \colon (1, \gamma^{-1}) \in e\}) = 1,$$

so, given $\epsilon > 0$, let N be large enough so that for $n \geq N$,

$$\sigma_n(\{e \colon [1]_e \neq [\gamma^{-1}]_e\}) < \epsilon.$$

Then

$$\left| \int \left(\frac{|A \cap [1]_e|}{|[1]_e|} - \frac{|A \cap [\gamma^{-1}]_e|}{|[\gamma^{-1}]_e|} \right) d\sigma_n(e) \right| \leq \epsilon,$$

and the proof is complete. □

15.3 Bauer vs Poulsen

The space $\mathrm{IRE}(\Gamma)$ is a Choquet simplex (being the space of invariant Borel probability measures for a continuous action of Γ on a compact metrizable space). Its extremal points are the ergodic IRE, whose set we denote by $\mathrm{ERGIRE}(\Gamma)$. We next consider the question of whether $\mathrm{IRE}(\Gamma)$ is a Bauer simplex, i.e., $\mathrm{ERGIRE}(\Gamma)$ is closed in $\mathrm{IRE}(\Gamma)$, or the Poulsen simplex, i.e., $\mathrm{ERGIRE}(\Gamma)$ is dense in $\mathrm{IRE}(\Gamma)$. By the results in Glasner-Weiss [GW], if Γ has property (T), then $\mathrm{IRE}(\Gamma)$ is a Bauer simplex. However the following is open:

Problem 15.8. *Assume that the countable group Γ does not have property* (T). *Is* $\mathrm{IRE}(\Gamma)$ *the Poulsen simplex?*

15.4 Another approach to the topology of equivalence relations

One can use ideas similar to those in this section to provide one more description of the topology of $S(E)$.

Fix $a \in A(\Gamma, X, \mu)$ with $E = E_a$. Consider the compact metrizable space $\mathcal{P}(\Gamma)^{\mathbb{N}} \times \mathrm{Eq}(\Gamma)$ (where $\mathcal{P}(\Gamma)$ is the space of all subsets of Γ, identified with 2^{Γ}), on which Γ acts continuously by $\gamma \cdot ((a_n), e) = ((\gamma a_n), \gamma \cdot e)$. Fix also a sequence (D_n) of Borel sets which is dense in MALG_μ. Define then the map

$$\theta_F^a = \theta_F \colon X \to \mathcal{P}(\Gamma)^{\mathbb{N}} \times \mathrm{Eq}(\Gamma),$$

by $\theta_F(x) = ((a_n), e)$, where $a_n = \{\gamma \colon \gamma^{-1} \cdot x \in D_n\}$ and $e = e_F(x)$. Let $\tau^a(F) = \tau(F) = (\theta_F)_* \mu \in \mathrm{Prob}(\mathcal{P}(\Gamma)^{\mathbb{N}} \times \mathrm{Eq}(\Gamma))$, the space of Borel probability measures on $(\mathcal{P}(\Gamma)^{\mathbb{N}} \times \mathrm{Eq}(\Gamma))$. Then $\tau(F)$ is Γ-invariant and so its projection on $\mathrm{Eq}(\Gamma)$ is in $\mathrm{IRE}(\Gamma)$.

Proposition 15.9. *The map $\tau^a \colon S(E) \to \mathrm{IRE}(\Gamma)$ is a homeomorphism into* $\mathrm{IRE}(\Gamma)$.

Proof. The continuity of τ^a is proved as in Proposition 15.1. That τ^a is injective follows from the paragraph preceding Proposition 4.14 and Lemma 4.10. That $(\tau^a)^{-1}$ is continuous can be deduced from the paragraph following Proposition 4.14. □

Thus we can also view $S(E)$ as a G_δ subset of $\mathrm{Prob}(\mathcal{P}(\Gamma)^{\mathbb{N}} \times \mathrm{Eq}(\Gamma))$.

15.5 Invariant, random equivalence relations and weak containment

Recall that for $a, b \in A(\Gamma, X, \mu)$, we let $a \preceq b$ iff a is **weakly contained** in b (see [K], where $\prec$ is used instead of $\preceq$). Concerning the map $\sigma^a(F)$ that sends $F \in S(E_a), a \in A(\Gamma, X, \mu)$, to an IRE on Γ, we consider its "slice" corresponding to the $\preceq$-predecessors of an action b.

Theorem 15.10. *Let Γ be an infinite countable group and $b \in A(\Gamma, X, \mu)$. Then the set*

$$\{\sigma^a(F) \colon a \in A(\Gamma, X, \mu), a \preceq b, F \in S(E_a)\}$$

is a compact subset of $\mathrm{IRE}(\Gamma)$.

Proof. We use the method of ultraproducts.

Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $a_n \in A(\Gamma, X, \mu), a_n \preceq b$ and $F_n \in S(E_{a_n})$, for $n \in \mathbb{N}$. As in the proof of Proposition 15.1, for each action $d \in A(\Gamma, Z, \rho)$, $F \in S(E_d)$, and $\alpha_i, \beta_i, \gamma_j, \delta_j \in \Gamma, i \leq m, j \leq k$, we put

$$A^d_{\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta},F} = \bigcap_{i \leq m} A^d_{(\alpha_i^d)^{-1},(\beta_i^d)^{-1},F} \cap \bigcap_{j \leq k} (X \setminus A^d_{(\gamma_j^d)^{-1},(\delta_j^d)^{-1},F}),$$

where for each $S, T \in [E_d], A^d_{S,T,F} = \{z \in Z \colon (S(z), T(z)) \in F\}$. In particular, $A^d_{T,F} = \{z \colon (z, T(z)) \in F\} = A_{id,T,F}$ and $A^d_{S,T,F} = S^{-1}(A^d_{TS^{-1},F})$.

We will show that there is a standard probability space (Y, ν), an action $c \in A(\Gamma, Y, \nu), c \preceq b$, and an equivalence relation $F \in S(E_c)$ on (Y, ν) such that

$$\nu(A^c_{\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta},F}) = \lim_{n \to \mathcal{U}} \mu(A^{a_n}_{\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta},F_n}),$$

for all $\alpha_i, \beta_i, \gamma_j, \delta_j \in \Gamma, i \leq m, j \leq k$, which implies that $\{\sigma^a(F) \colon a \in A(\Gamma, X, \mu), a \preceq b, F \in S(E_a)\}$ is compact in $\mathrm{IRE}(\Gamma)$.

We will use below the notation and terminology of Conley–Kechris–Tucker-Drob [CKT] concerning ultraproducts. Let $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$ be the ultrapower of (X, μ) and let $a = \prod_n a_n / \mathcal{U}$ the ultraproduct of (a_n). Put for $g \in \Gamma$,

$$A^{a_n}_{g,F_n} = \{x \in X \colon (x, g^{a_n}(x)) \in F_n\}.$$

Then for each n, $(A^{a_n}_{g,F_n})$ satisfies conditions 1.-4. of Lemma 4.12. So if $A_g = [(A^{a_n}_{g,F_n})]_{\mathcal{U}}$ is the ultrapower of $(A^{a_n}_{g,F_n})$, it follows that $(A_g)_{g \in \Gamma}$ also satisfies these conditions (all of course $\mu_{\mathcal{U}}$-a.e.).

If $\boldsymbol{B}_{\mathcal{U}}$ is the σ-algebra on which $\mu_{\mathcal{U}}$ lives, let $\mathrm{MALG}_{\mu_{\mathcal{U}}}$ be the measure algebra of $(X_{\mathcal{U}}, \boldsymbol{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$. By the proof of Proposition 4.3 in [CKT], there is a map $T_{\mathcal{U}} \colon \Gamma \times \mathrm{MALG}_{\mu_{\mathcal{U}}} \to \mathrm{MALG}_{\mu_{\mathcal{U}}}$ such that if $g \in \Gamma, A \in \mathrm{MALG}_{\mu_{\mathcal{U}}} \setminus \{\emptyset\}$ and $g^a(x) \neq x, \forall x \in A$, then $T_{\mathcal{U}}(g, A) \subseteq A, \mu_{\mathcal{U}}(T_{\mathcal{U}}(g, A)) \geq \frac{1}{16}\mu_{\mathcal{U}}(A)$ and $g^a \cdot T_{\mathcal{U}}(g, A) \cap T_{\mathcal{U}}(g, A) = \emptyset$.

As in [CKT, Sections 4.2, 4.3], fix a countable Boolean subalgebra $\boldsymbol{B}_0 \subseteq \mathrm{MALG}_{\mu_{\mathcal{U}}}$ which contains all $A_g, \mathrm{Fix}(g^a), g \in \Gamma$, and is closed under the action a, the function $S_{\mathcal{U}}$ in [CKT, Section 3.2] and the function $T_{\mathcal{U}}$ as above. Let $\boldsymbol{B} = \sigma(\boldsymbol{B}_0) \subseteq \mathrm{MALG}_{\mu_{\mathcal{U}}}$ be the σ-algebra generated by $\boldsymbol{B}_0$. This is a countably generated, non-atomic, a-invariant subalgebra of $\mathrm{MALG}_{\mu_{\mathcal{U}}}$, so there is a standard probability space (Y, ν) and a measurable map $\pi \colon X_{\mathcal{U}} \to Y$ with $\pi_* \mu_{\mathcal{U}} = \nu$ and an action $c \in A(\Gamma, Y, \nu)$ such that

$$\pi(g^a(x)) = g^c(\pi(x)), g \in \Gamma, x \in X_{\mathcal{U}}$$

(i.e., c is a factor of a) and $B \mapsto \pi^{-1}(B)$ is an isomorphism of (MALG_ν, ν) with $(\boldsymbol{B}, \mu_{\mathcal{U}} | \boldsymbol{B})$ preserving the Γ-action.

Let then $B_g, g \in \Gamma$, in MALG_ν, be such that $\pi^{-1}(B_g) = A_g$. Then $\nu(B_g) = \mu_{\mathcal{U}}(A_g)$ and the family $(B_g)_{g \in \Gamma}$ satisfies 1.-3. of Lemma 4.12. We will next verify that condition 4. of the same proposition also holds. Assuming this, there will be an equivalence relation F on (Y, ν) with $A^c_{g,F} = B_g$. Replacing F by $F \cap E_c$, we can assume that $F \subseteq E_c$. Then, for each $\alpha_i, \beta_i, \gamma_j, \delta_j \in \Gamma, i \leq m, j \leq k$,

$$\nu(A^c_{\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, F}) = \lim_{n \to \mathcal{U}} \mu(A^{a_n}_{\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, F_n}).$$

Since c is a factor of an ultraproduct of (a_n) and $a_n \preceq b$, for each n, then $c \preceq b$ (see [CKT, Theorem 1]) and the proof is complete.

In order to verify condition 4. in Lemma 4.12, it is enough to show that for each $g \in \Gamma$,

$$\pi^{-1}(\mathrm{Fix}(g^c)) = \mathrm{Fix}(g^a).$$

It is clear that $\pi^{-1}(\mathrm{Fix}(g^c)) \supseteq \mathrm{Fix}(g^a)$. If they are distinct (in $\mathrm{MALG}_{\mu_{\mathcal{U}}}$), let $A = \pi^{-1}(\mathrm{Fix}(g^c)) \setminus \mathrm{Fix}(g^a) \in \boldsymbol{B}$ and let $\mu_{\mathcal{U}}(A) = \epsilon > 0$. Then $g^a(x) \neq x, \forall x \in A$. Let $B \in \boldsymbol{B}_0$ be such that $\mu_{\mathcal{U}}(B \triangle A) < \frac{\epsilon}{32}$. Since $A \subseteq X_{\mathcal{U}} \setminus \mathrm{Fix}(g^a) \in \boldsymbol{B}_0$, we can assume (by replacing B by $B \setminus \mathrm{Fix}(g^a)$) that $B \cap \mathrm{Fix}(g^a) = \emptyset$. Since $\boldsymbol{B}_0$ is closed under $T_{\mathcal{U}}$, let $C \subseteq B, C \in \boldsymbol{B}_0$ be such that $g^a \cdot C \cap C = \emptyset$ and $\mu_{\mathcal{U}}(C) \geq \frac{1}{16}\mu_{\mathcal{U}}(B)$. In particular $C \cap A \neq \emptyset$. Since $C \cap A \in \boldsymbol{B}$, let $D \in \mathrm{MALG}_\nu$ be such that $\pi^{-1}(D) = C \cap A \subseteq \pi^{-1}(\mathrm{Fix}(g^c))$, so $\emptyset \neq D \subseteq \mathrm{Fix}(g^c)$. On the other hand, $\pi^{-1}(g^c \cdot D) = g^a \cdot (C \cap A)$, so $\pi^{-1}(g^c \cdot D) \cap \pi^{-1}(D) = g^a \cdot (C \cap A) \cap (C \cap A) = \emptyset$, so $g^c \cdot D \cap D = \emptyset$, while $g^c \cdot D = D$, a contradiction. □

Corollary 15.11. *Let* Γ *be an infinite countable group and assume that* $b \in A(\Gamma, X, \mu)$ *is ergodic but not strongly ergodic. Then the set*

$$\{\sigma^a(F) \colon a \in A(\Gamma, X, \mu), a \preceq b, F \in S(E_a)\}$$

is a compact convex subset of $\mathrm{IRE}(\Gamma)$*.*

Proof. By [AW, Theorem 3] the set $\{a \in A(\Gamma, X, \mu) \colon a \preceq b\}$ is closed under convex combinations (see [K, Section 10, **(F)**] for the concept of convex combinations of actions). □

16. Ultraproducts of equivalence relations

We will use here again the notation of Section 15.5 and [CKT]. Consider the space (X, μ) and for each non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ form the ultrapower $X_\mathcal{U}$ with the associated measure $\mu_\mathcal{U}$. For $(x_n) \in X^\mathbb{N}$, put $[x_n]_\mathcal{U} = [(x_n)]_\mathcal{U} \in X_\mathcal{U}$. We will use below the following general fact, where for Borel $A \subseteq X$, we put $[A]_\mathcal{U} = \{[x_n]_\mathcal{U} \in X_\mathcal{U} \colon \mathcal{U}n(x_n \in A)\} \subseteq X_\mathcal{U}$.

Proposition 16.1. $[\bigcup_{i\in\mathbb{N}} A_i]_\mathcal{U} = \bigcup_{i\in\mathbb{N}}[A_i]_\mathcal{U}$ *in* $\mathrm{MALG}_{\mu_\mathcal{U}}$.

Proof. Let $B_j = \bigcup_{i\leq j} A_i$. Then $[B_j]_\mathcal{U} = \bigcup_{i\leq j}[A_i]_\mathcal{U}$ and $\bigcup_j B_j = \bigcup_i A_i$, $\bigcup_j [B_j]_\mathcal{U} = \bigcup_i [A_i]_\mathcal{U}$, so we can assume that $A_0 \subseteq A_1 \subseteq \dots$, and thus $[A_0]_\mathcal{U} \subseteq [A_1]_\mathcal{U} \subseteq \cdots \subseteq [\bigcup_i A_i]_\mathcal{U}$. Let $\mu_\mathcal{U}([\bigcup_i A_i]_\mathcal{U}) = t$. It is enough to show that $\mu_\mathcal{U}(\bigcup_i [A_i]_\mathcal{U}) = t$. Now $t = \mu(\bigcup_i A_i) = \lim_{i\to\infty}\mu(A_i)$ and thus $\mu(\bigcup_i [A_i]_\mathcal{U}) = \lim_{i\to\infty}\mu([A_i]_\mathcal{U}) = t$. $\square$

Consider now a sequence of measure preserving countable Borel equivalence relations (F_n) on (X,μ). Let $E \in \mathcal{E}$ be such that $F_n \subseteq E$, for each n. Fix an action $a \in A(\Gamma, X, \mu)$ such that $E_a = E$. We will use this to define an ultraproduct $\prod_n^a F_n/\mathcal{U}$ of the F_n. We will then show that it is independent of E and the action a, so that we can define unambiguously the ultraproduct $\prod_n F_n/\mathcal{U}$.

As in the proof of Theorem 15.10, we let $A^a_{g,F_n} = \{x \in X \colon (x, g^a(x)) \in F_n\}$ and $A^a_g = [(A^a_{g,F_n})]_\mathcal{U}$. Consider also the ultrapower $a_\mathcal{U} = \prod_n a/\mathcal{U}$. Then (A^a_g) satisfies conditions 1.-4. of Lemma 4.12 and therefore it gives rise to a countable equivalence relation $\hat{F}^a = \prod_n^a F_n/\mathcal{U}$ on $X_\mathcal{U}$ defined by

$$[x_n]_\mathcal{U} \hat{F}^a [y_n]_\mathcal{U} \iff \exists g \in \Gamma(g^{a_\mathcal{U}}([x_n]_\mathcal{U}) = [y_n]_\mathcal{U} \ \&\ [x_n]_\mathcal{U} \in A^a_g).$$

Thus $\prod_n^a F_n/\mathcal{U}$ is the union of the graphs of $g^{a_\mathcal{U}}|A^a_g$, $g \in \Gamma$. It is easy to see that the equivalence relation induced by each $g^{a_\mathcal{U}}|A^a_g$ is also induced by a single measure preserving automorphism of $(X_\mathcal{U}, \mu_\mathcal{U})$ and thus $\prod_n^a F_n/\mathcal{U}$ is induced by a measure preserving action of a countable group on $(X_\mathcal{U}, \mu_\mathcal{U})$. Thus we can view $\prod_n^a F_n/\mathcal{U}$ as a countable, measure preserving equivalence relation on $(X_\mathcal{U}, \mu_\mathcal{U})$. Note that we also have $A^{a_\mathcal{U}}_{g,\hat{F}^a} = [(A^{a_n}_{g,F_n})]_\mathcal{U}$ and so $\mu_\mathcal{U}(A^{a_\mathcal{U}}_{g,\hat{F}^a}) = \lim_{n\to\mathcal{U}} \mu(A^{a_n}_{g,F_n})$.

We now check that this construction is independent of E, a. Suppose $F_n \subseteq E \subseteq F$ for each n and let $a \in A(\Gamma, X, \mu)$ generate E and $b \in A(\Delta, X, \mu)$ generate F. We will show that $\hat{F}^a = \prod_n^a F_n/\mathcal{U} = \prod_n^b F_n/\mathcal{U} = \hat{F}^b$.

(i) Suppose $[x_n]_\mathcal{U} \hat{F}^a [y_n]_\mathcal{U}$ and find $g \in \Gamma$ with $g^{a_\mathcal{U}}([x_n]_\mathcal{U}) = [y_n]_\mathcal{U}$ and $[x_n]_\mathcal{U} \in A^a_g$, i.e., $\mathcal{U}n((x_n, g^a(x_n)) \in F_n)$. Write $X = \bigsqcup_{d\in\Delta} X_d$, where X_d is Borel and

$$x \in X_d \Rightarrow g^a(x) = d^b(x)$$

(since $E = E_a \subseteq E_b = F$). Then $X_\mathcal{U} = \bigsqcup_{d\in\Delta}[X_d]_\mathcal{U}$, so $[x_n]_\mathcal{U} \in [X_d]_\mathcal{U}$ for some $d \in \Delta$ and therefore $\mathcal{U}n(x_n \in X_d)$, so that $\mathcal{U}n(g^a(x_n) = d^b(x_n))$ and thus $g^{a_\mathcal{U}}([x_n]_\mathcal{U}) = [y_n]_\mathcal{U} = d^{b_\mathcal{U}}([x_n]_\mathcal{U})$. Moreover, $\mathcal{U}n((x_n, d^b(x_n)) \in F_n))$, i.e., $[x_n]_\mathcal{U} \in A^b_d$, so $[x_n]_\mathcal{U} \hat{F}^b [y_n]_\mathcal{U}$.

(ii) Conversely assume that $[x_n]_\mathcal{U} \hat{F}^b [y_n]_\mathcal{U}$ and find $d \in \Delta$ with $d^{b_\mathcal{U}}([x_n]_\mathcal{U}) = [y_n]_\mathcal{U}$ and $\mathcal{U}n((x_n, d^b(x_n)) \in F_n)$. By Proposition 4.2, there is $T \in [E]$ such that

$$(x, d^b(x)) \in E \Rightarrow d^b(x) = T(x).$$

Let then $X = \bigsqcup_{g\in\Gamma} X_g$ be a Borel decomposition such that

$$x \in X_g \Rightarrow T(x) = g^a(x),$$

so that

$$x \in X_g \ \&\ (x, d^b(x)) \in E \Rightarrow d^b(x) = g^a(x).$$

Now $[x_n]_\mathcal{U} \in [X_g]_\mathcal{U}$ for some $g \in \Gamma$, i.e., $\mathcal{U}n(x_n \in X_g)$. But also

$$\mathcal{U}n((x_n, d^b(x_n)) \in F_n \subseteq E),$$

so $\mathcal{U}n(d^b(x_n) = g^a(x_n))$, i.e., $d^{b_\mathcal{U}}([x_n]_\mathcal{U}) = [y_n]_\mathcal{U} = g^{a_\mathcal{U}}([x_n]_\mathcal{U})$ and moreover $\mathcal{U}n((x_n, g^a(x_n) \in F_n)$, so $[x_n]_\mathcal{U} \hat{F}^a [y_n]_\mathcal{U}$.

17. Factors

We will discuss here various notions of factoring for equivalence relations and their applications.

17.1 Factors in general

Let E be a measure preserving countable Borel equivalence relation on (X, μ). Let $\mathcal{A} \subseteq \mathrm{MALG} = \mathrm{MALG}_\mu$ be a non-atomic, σ-subalgebra of MALG. Put

$$[E]^{\mathcal{A}} = \{T \in [E] \colon \forall A \in \mathcal{A}(T(A), T^{-1}(A) \in \mathcal{A})\}.$$

This is a closed subgroup of $([E], u)$, which we call the **relative to $\mathcal{A}$ full group of** E.

Consider now a separable subgroup Γ of $(\mathrm{Aut}(X, \mu), u)$. This defines a measure preserving countable Borel equivalence relation F^Γ as follows: Let $\Gamma_0 \leq \Gamma$ be a countable dense subgroup of Γ and let F^Γ be the equivalence relation induced by Γ_0. We can easily see that this is independent of the choice of Γ_0 and moreover $\Gamma \leq [F^\Gamma]$.

Clearly F^Γ is the smallest equivalence relation F such that $\Gamma \leq [F]$. Kittrell-Tsankov [KT, 4.14] have shown that if Γ is also closed in the uniform topology, then there is a largest equivalence relation F, denoted by F_Γ, such that $[F] \leq \Gamma$ and moreover $[F_\Gamma]$ is a normal subgroup of Γ.

We now say that E is **generated by** $\mathcal{A}$ or that $\mathcal{A}$ **generates** E if $F^{[E]^{\mathcal{A}}} = E$ (clearly always $F^{[E]^{\mathcal{A}}} \subseteq E$). This is equivalent to saying that there is a countable group Γ and an action $a \in A(\Gamma, X, \mu)$ such that $E_a = E$ and $\mathcal{A}$ is invariant under a, i.e., for each $A \in \mathcal{A}, g \in \Gamma$ we have that $g^a(A) \in \mathcal{A}$.

Let now $\pi \colon (X, \mu) \to (Y, \nu)$ be the factor corresponding to $\mathcal{A}$, so that (Y, ν) is a standard (non-atomic) measure space, $\pi_* \mu = \nu$ and $B \mapsto \pi^{-1}(B)$

is an isomorphism of (MALG_ν, ν) with $(\mathcal{A}, \mu|\mathcal{A})$ (see [K2, 17.43]). If $T \in \mathrm{Aut}(X, \mu)$ preserves $\mathcal{A}$ (i.e., $\forall A \in \mathcal{A}(T(A), T^{-1}(A) \in \mathcal{A})$), then (via π^{-1}) it gives an automorphism of MALG_ν, i.e., an element of $\mathrm{Aut}(Y, \nu)$, denoted by $\hat{\pi}(T)$, such that $\hat{\pi}(T)(\pi(x)) = \pi(T(x))$. (To verify this equality, simply check that for every $B \in \mathrm{MALG}_\nu$, $\hat{\pi}(T)(\pi(x)) \in B \iff \pi(T(x)) \in B$.) In particular, $\pi(x) = \pi(y) \implies \pi(T(x)) = \pi(T(y))$. So if

$$\mathrm{Aut}(X, \mu)^{\mathcal{A}} = \{T \in \mathrm{Aut}(X, \mu) \colon \forall A \in \mathcal{A}(T(A), T^{-1}(A) \in \mathcal{A})\},$$

then $\mathrm{Aut}(X, \mu)^{\mathcal{A}}$ is a closed subgroup of $(\mathrm{Aut}(X, \mu), u)$ and

$$\hat{\pi} \colon (\mathrm{Aut}(X, \mu)^{\mathcal{A}}, u) \to (\mathrm{Aut}(Y, \nu), u)$$

is a continuous homomorphism. In particular, $\hat{\pi}([E]^{\mathcal{A}})$ is a separable subgroup of $(\mathrm{Aut}(Y, \nu), u)$ and thus gives rise to the equivalence relation $F = F^{\hat{\pi}([E]^{\mathcal{A}})}$. We call this **the factor of E relative to $\mathcal{A}$**.

Note that if $\Gamma_0 \leq [E]^{\mathcal{A}}$ is dense in $([E]^{\mathcal{A}}, u)$, so that Γ_0 generates E, then $\hat{\pi}(\Gamma_0)$ is dense in $(\hat{\pi}([E]^{\mathcal{A}}), u)$ and so, by definition, it generates the factor F. It follows that there is a countable group Γ and an action $a \in A(\Gamma, X, \mu)$ preserving $\mathcal{A}$ with $E_a = E$ such that if $\hat{\pi}(a) = b$ is the factor action of a via π (i.e., $g^b = \hat{\pi}(g^a)$ for each $g \in \Gamma$), so that

$$\pi(g^a(x)) = g^b(\pi(x)),$$

then we have $E_b = F$. Therefore π is a **homomorphism** of E into F, i.e.,

$$xEy \Rightarrow \pi(x)F\pi(y)$$

and also π is **class-surjective**, i.e., the image of each E-class is an F-class. Moreover if $c \in A(\Delta, X, \mu)$ is *any* action of a countable group Δ preserving $\mathcal{A}$ with $E_c = E$ and $\hat{\pi}(c) = d$ is the factor action of c via π, then $E_d = F$. Indeed, let yFz and choose x with $\pi(x) = y$ and $g \in \Gamma$ with $g^b(y) = z$. Then $g^a(x) = h^c(x)$ for some $h \in \Delta$, since $E_a = E_c$, so $\pi(g^a(x)) = g^b(\pi(x)) = g^b(y) = z = \pi(h^c(x)) = h^d(\pi(x)) = h^d(y)$, so $(y, z) \in E_d$. Thus $F \subseteq E_d$. Since obviously $E_d \subseteq F$, we are done.

Clearly $\hat{\pi}$ is a homomorphism of $[E]^{\mathcal{A}}$ into $[F]$. In fact we have:

Proposition 17.1. *The homomorphism $\hat{\pi} \colon [E]^{\mathcal{A}} \to [F]$ is surjective.*

Proof. Let $S \in [F]$. Let $a \in A(\Gamma, X, \mu), E_a = E, \hat{\pi}(a) = b, E_b = F$ as before. Then there is a Borel decomposition $Y = \bigsqcup_{g \in \Gamma} Y_g$ such that

$$y \in Y_g \Rightarrow S(y) = g^b(y).$$

Let $X_g = \pi^{-1}(Y_g) \in \mathcal{A}$, so that $X = \bigsqcup_{g \in \Gamma} X_g$. If $g, h \in \Gamma$ are distinct, then $g^b(Y_g) \cap h^b(Y_h) = S(Y_g) \cap S(Y_h) = \emptyset$ and $\bigsqcup_{g \in \Gamma} g^b(Y_g) = \bigsqcup_{g \in \Gamma} S(Y_g) = Y$, so that $g^a(X_g) \cap h^a(X_h) = \emptyset$ and $X = \bigsqcup_{g \in \Gamma} g^a(X_g)$. Put $T = \bigsqcup_{g \in \Gamma} g^a | X_g$.

First note that $T \in [E]^{\mathcal{A}}$, since if $A \in \mathcal{A}$, then $T(A) = T(\bigsqcup_{g \in \Gamma}(A \cap X_g)) = \bigsqcup_{g \in \Gamma} g^a(A \cap X_g) \in \mathcal{A}$. We will finally verify that $\hat{\pi}(T) = S$. For that it is enough to check that for each $B \in \mathrm{MALG}_\nu, g \in \Gamma$ we have that $\hat{\pi}(T)(B \cap Y_g) = S(B \cap Y_g)$. This is the case, since $\hat{\pi}(T)(B \cap Y_g) = \pi(T(\pi^{-1}(B) \cap X_g)) = \pi(g^a(\pi^{-1}(B) \cap X_g)) = g^b(B \cap Y_g) = S(B \cap Y_g)$. □

The kernel of $\hat{\pi}|[E]^{\mathcal{A}}$ is equal to

$$[E]_{\mathcal{A}} = \{T \in [E]^{\mathcal{A}} \colon \forall A \in \mathcal{A}(T(A) = A)\},$$

thus $[F] \cong [E]^{\mathcal{A}}/[E]_{\mathcal{A}}$ (as topological groups). Note also that $T \in [E]_{\mathcal{A}} \iff T \in [E]^{\mathcal{A}} \wedge \pi(T(x)) = \pi(x), \forall x$.

Let R_π be the **kernel** of π, i.e., the smooth equivalence relation given by:

$$x R_\pi y \iff \pi(x) = \pi(y).$$

Put also

$$E_\pi = E \cap R_\pi.$$

Thus $[E]_{\mathcal{A}} = [E_\pi]$.

It is easy to check that E, R_π **commute**, i.e., $E \circ R_\pi = R_\pi \circ E$. (Here for any two equivalence relations E_1, E_2, we define the relation $E_1 \circ E_2$ by $x \; E_1 \circ E_2 \; y \iff \exists z(x E_1 z \wedge z E_2 y)$.)

We now have:

Proposition 17.2. *Let F be a factor of E, let $S_0, S_1, \dots \in [F]$ be such that $F = E_{S_0, S_1, \dots}$, and let $T_0, T_1, \dots \in [E]^{\mathcal{A}}$ be such that $\hat{\pi}(T_i) = S_i$. If $E' = E_{T_0, T_1, \dots}$, then $E = E' \vee E_\pi$.*

Proof. Let xEy. Then $\pi(x) F \pi(y)$, so for some $i_1, \dots, i_n$ we have $\pi(y) = S_{i_1}^{\pm 1} \circ \dots \circ S_{i_n}^{\pm 1}(\pi(x))$. Then if $z = T_{i_1}^{\pm 1} \circ \dots \circ T_{i_n}^{\pm 1}(x)$, we have $\pi(z) = S_{i_1}^{\pm 1} \circ \dots \circ S_{i_n}^{\pm 1}(\pi(x)) = \pi(y)$, so $x E' z E_\pi y$. □

The following result was shown by R. Tucker-Drob.

Proposition 17.3 (Tucker-Drob). *Let $S \in [F]$ be an involution. Then there is an involution $T \in [E]^{\mathcal{A}}$ with $\hat{\pi}(T) = S$.*

Proof. By Proposition 17.1, let $\tilde{T} \in [E]^{\mathcal{A}}$ be such that $\hat{\pi}(\tilde{T}) = S$. We can define $T(x) = x$ for all x such that $S(\pi(x)) = \pi(x)$, so that working in the complement of the set of such x's, we can assume that $S(\pi(x)) \neq \pi(x)$, for all x. Let $\Phi = \{\{x, x'\}\colon xEx' \wedge S(\pi(x)) = \pi(x')\}$. Then, by [KM, Lemma 7.3], we can find a Borel set $A \subseteq X$ and a Borel equivalence relation R on A such that $[x]_R \in \Phi$, for $x \in A$ and if $\{x, x'\} \cap A = \emptyset$, then $\{x, x'\} \notin \Phi$.

For $x \in A$, we can define $T(x) = x'$, where $\{x, x'\} \in R$. Clearly $\pi(T(x)) = S(\pi(x))$, so if we can show that $A = X$ (modulo null sets), this will imply that $T \in [E]^{\mathcal{A}}$, T is an involution and $\hat{\pi}(T) = S$.

Let $B = \{x\colon \forall x'((xEx' \wedge \pi(x) = \pi(x')) \implies x' \in A) \subseteq A$. Then by the properties of A, R, we have that $x \notin A \implies \tilde{T}(x) \in B$ (else there would be some x' such that $x' \notin A$ and $\{x, x'\} \in \Phi$.) Also $(X \setminus A) \cap T(B) \subseteq (X \setminus A) \cap A = \emptyset$ and $T(B) \subseteq \tilde{T}^{-1}(B)$. Therefore $X \setminus A \subseteq \tilde{T}^{-1}(B) \setminus T(B)$ and since $\mu(\tilde{T}^{-1}(B)) = \mu(T(B)) = \mu(B)$, $X \setminus A$ is null. □

Corollary 17.4. *If E is generated by the σ-subalgebra $\mathcal{A}$, then there are involutions $T_0, T_1, \dots \in [E]^{\mathcal{A}}$ such that $E = E_{T_0, T_1, \dots}$.*

Proof. Let $S_0, S_1, \dots \in [F]$ be involutions such that $F = E_{S_0, S_1, \dots}$. By Proposition 17.3, let $U_0, U_1, \dots \in [E]^{\mathcal{A}}$ be involutions such that $\hat{\pi}(U_i) = S_i$. Let $E' = E_{U_0, U_1, \dots}$. Then, by Proposition 17.2, $E = E' \vee E_\pi$.

Now let $V_0, V_1, \dots$ be involutions in $[E_\pi]$ such that $E_\pi = E_{V_0, V_1, \dots}$. Clearly $V_0, V_1, \dots \in [E]^{\mathcal{A}}$ and so if $\{T_0, T_1, \dots\} = \{U_0, U_1, \dots\} \cup \{V_0, V_1, \dots\}$, then $T_0, T_1, \dots \in [E]^{\mathcal{A}}$ and $E = E_{T_0, T_1, \dots}$. □

We next show the following.

Theorem 17.5. *The composition of factors is a factor.*

Proof. Let E live on (X, μ), $\pi\colon (X, \mu) \to (Y, \nu)$ be the factor corresponding to the σ-subalgebra $\mathcal{A} \subseteq \mathrm{MALG}_\mu$ which generates E and let F be the corresponding factor. Let also $\mathcal{B}$ be a σ-subalgebra of MALG_ν such that F is generated by $\mathcal{B}$ and let $\rho\colon (Y, \nu) \to (Z, \omega)$ and H be the factor equivalence relation corresponding to $\mathcal{B}$. Let $\sigma = \rho \circ \pi\colon (X, \mu) \to (Z, \omega)$ be the composition with associated σ-subalgebra $\mathcal{C} = \pi^{-1}(\mathcal{B}) \subseteq \mathcal{A}$. We will show that H is the factor of E corresponding to $\mathcal{C}$.

Lemma 17.6. $[F]^{\mathcal{B}} = \hat{\pi}([E]^{\mathcal{C}} \cap [E]^{\mathcal{A}})$.

Proof. Since $\hat{\pi}([E]^{\mathcal{A}}) = [F]$, this is clear from the definitions noting that if $T \in [E]^{\mathcal{A}}$, then $T \in [E]^{\mathcal{C}}$ iff $\hat{\pi}(T) \in [F]^{\mathcal{B}}$. $\square$

Lemma 17.7. $E = F^{([E]^{\mathcal{C}} \cap [E]^{\mathcal{A}})}$ *(in particular* $E = F^{[E]^{\mathcal{C}}}$*).*

Proof. Let $T \in [E]^{\mathcal{A}}$. Then $\hat{\pi}(T) \in [F]$, so, since $F = F^{[F]^{\mathcal{B}}}$, we can find S_i in $[F]^{\mathcal{B}}$ and disjoint Borel sets $Y_i \subseteq Y$ with $\bigsqcup_i Y_i = Y$ such that $\hat{\pi}(T) = \bigsqcup_i S_i|Y_i$. By Lemma 17.6, let $T_i \in [E]^{\mathcal{C}} \cap [E]^{\mathcal{A}}$ be such that $\hat{\pi}(T_i) = S_i$, so that $\hat{\pi}(T) = \bigsqcup_i \hat{\pi}(T_i)|Y_i$. Then for each i, $\hat{\pi}(T)|Y_i = \hat{\pi}(T_i)|Y_i$ or $\hat{\pi}(T_i^{-1}T)|Y_i = id|Y_i$.

Let $X_i = \pi^{-1}(Y_i) \in \mathcal{A}$. It follows that $T_i^{-1}T(A) = A$ for any $A \in \mathcal{A}, A \subseteq X_i$ and in particular $T_i^{-1}T(X_i) = X_i$. Since $X = \bigsqcup_i X_i, U = \bigsqcup_i (T_i^{-1}T)|X_i \in [E]^{\mathcal{A}}$. Moreover $U(A) = A$ for every $A \in \mathcal{A}$, so that actually $U \in [E]_{\mathcal{A}} \subseteq [E]^{\mathcal{C}}$. Now for each x, there is i such that $T_i^{-1}T(x) = U(x)$ or $T(x) = T_iU(x)$. Since $T_iU \in [E]^{\mathcal{C}} \cap [E]^{\mathcal{A}}$, we have that $T \in [F^{[E]^{\mathcal{C}} \cap [E]^{\mathcal{A}}}]$. But $E = F^{[E]^{\mathcal{A}}}$, so $E = F^{[E]^{\mathcal{C}} \cap [E]^{\mathcal{A}}}$. $\square$

We now complete the proof of Theorem 17.5 as follows. Let $\Gamma_0 \leq [E]^{\mathcal{C}} \cap [E]^{\mathcal{A}}$ be a countable dense subgroup of $[E]^{\mathcal{C}} \cap [E]^{\mathcal{A}}$, which therefore generates E. Then $\hat{\pi}(\Gamma_0)$ is a dense subgroup of $[E]^{\mathcal{B}}$, so $\hat{\rho} \circ \hat{\pi}(\Gamma_0) = \hat{\sigma}(\Gamma_0)$ generates H. By the arguments preceding Proposition 17.1, it follows that H is the factor of E corresponding to $\mathcal{C}$. $\square$

It also follows from the preceding argument that there is a countable group Γ and an action $a \in A(\Gamma, X, \mu)$, preserving both $\mathcal{A}$ and $\mathcal{C}$, such that $E_a = E$, and moreover if $\hat{\pi}(a) = b$, then $E_b = F$ and b preserves $\mathcal{B}$ and if $\hat{\rho}(b) = \hat{\rho}(\hat{\pi}(a)) = c$, then $E_c = H$.

This can be extended to infinite chains as follows.

For each $n \in \mathbb{N}$, let E_n be an equivalence relation on (X_n, μ_n) and for each $n \geq 1$, let $\pi_n \colon (X_n, \mu_n) \to (X_{n-1}, \mu_{n-1})$ be the map corresponding to a σ-subalgebra $\mathcal{A}_n \subseteq \mathrm{MALG}_{\mu_n}$, which generates E_n, and let E_{n-1} be the factor corresponding to $\mathcal{A}_n$. For $n > m$, let $\pi_{n,m} = \pi_{m+1} \circ \cdots \circ \pi_{n-1} \circ \pi_n$, and let $\pi_{n,n}$ = identity on X_n. Then $\pi_{n,m} \colon X_n \to X_m$, for $n \geq m$. Put $\mathcal{A}_{n,m} = \pi_{n,m}^{-1}(\mathrm{MALG}_{\mu_m})$, so that $\mathcal{A}_{n,n-1} = \mathcal{A}_n$ and $\mathcal{A}_{n,n} = \mathrm{MALG}_{\mu_n}$. Thus we have the following σ-subalgebras of MALG_{μ_n},

$$\mathcal{A}_{n,0} \subseteq \mathcal{A}_{n,1} \subseteq \cdots \subseteq \mathcal{A}_{n,n-1} = \mathcal{A}_n \subseteq \mathcal{A}_{n,n} = \mathrm{MALG}_{\mu_n}.$$

Put

$$[E_n]^* = [E_n]^{\mathcal{A}_{n,0}} \cap \cdots \cap [E_n]^{\mathcal{A}_{n,n-1}}.$$

Then we have, generalizing Lemma 17.6, Lemma 17.7:

Proposition 17.8. *For each* $n \geq 1$,

(i) $\hat{\pi}_n([E_n]^*) = [E_{n-1}]^*$,

(ii) $E_n = F^{[E_n]^*}$.

Proof. By induction on $n \geq 1$. The case $n = 1$ is clear. So assume that (i), (ii) hold for $n - 1 \geq 1$ and prove them for n. First we will show that if we assume (i) for n, then (ii) also holds for n. The proof is similar to that of Lemma 17.7.

Let $T \in [E_n]^{\mathcal{A}_n}$. Then $\hat{\pi}_n(T) \in [E_{n-1}]$, so by (ii) for $n-1$, there is a sequence $S_i \in [E_{n-1}]^*$ and $Y_i \in \mathrm{MALG}_{\mu_{n-1}}$ such that $\bigsqcup_i Y_i = Y$ and $\hat{\pi}_n(T) = \bigsqcup_i (S_i|Y_i)$. Let then, using (i) for n, $T_i \in [E_n]^*$ be such that $\hat{\pi}_n(T_i) = S_i$. Then $\hat{\pi}_n(T) = \bigsqcup_i (\hat{\pi}_n(T_i)|Y_i)$, so $\hat{\pi}_n(T_i^{-1}T)|Y_i = id|Y_i$. Let $X_i = \pi_n^{-1}(Y_i) \in \mathcal{A}_n$. Then for any $A \subseteq X_i, A \in \mathcal{A}_n, T_i^{-1}T(A) = A$, so, in particular, $T_i^{-1}T(X_i) = X_i$. Since $X = \bigsqcup_i X_i$, we have that $U = \bigsqcup_i (T_i^{-1}T) \in [E]$. Also $U(A) = A$ for $A \in \mathcal{A}_n$, so $U \in [E_n]_{\mathcal{A}_n} \subseteq [E_n]^*$. Now for each $x \in X_n$, there is i such that $U(x) = T_i^{-1}T(x)$, i.e., $T(x) = T_iU(x)$. Since $T_iU \in [E_n]^*$, this shows that $T \in [F^{[E_n]^*}]$, thus $E_n \subseteq F^{[E_n]^*} \subseteq E_n$, so (ii) holds.

We now prove (i) for n. Clearly $\hat{\pi}_n([E_n]^*) \subseteq [E_{n-1}]^*$. Conversely, if $S \in [E_{n-1}]^*$, let $T \in [E_n]^{\mathcal{A}_n}$ be such that $\hat{\pi}_n(T) = S$. Since S keeps invariant the σ-subalgebras $\mathcal{A}_{n-1,0}, \ldots, \mathcal{A}_{n-1,n-2}$, clearly T keeps invariant

$$\mathcal{A}_{n,0} = \pi_n^{-1}(\mathcal{A}_{n-1,0}), \ldots, \mathcal{A}_{n,n-2} = \pi_n^{-1}(\mathcal{A}_{n-1,n-2}), \mathcal{A}_{n,n-1},$$

so $T \in [E_n]^*$. □

Consider now the inverse limit (X_∞, μ_∞) of the sequence $(X_n, \mu_n), \pi_n$. Denote by $\pi_{\infty,n} \colon (X_\infty, \mu_\infty) \to (X_n, \mu_n)$ the associated maps, so that $\pi_{n,m} \circ \pi_{\infty,n} = \pi_{\infty,m}$ for $n \geq m$. Thus X_∞ consist of all chains $(x_n) \in \prod_n X_n$ with $\pi_n(x_n) = x_{n-1}$, for $n \geq 1$, $\pi_{\infty,n}((x_n)) = x_n$ and $\mathrm{MALG}_{\mu_\infty}$ is the smallest σ-algebra containing the σ-subalgebras

$$\mathcal{A}_{\infty,0} = \pi_{\infty,0}^{-1}(\mathrm{MALG}_{\mu_0}) \subseteq \mathcal{A}_{\infty,1} = \pi_{\infty,1}^{-1}(\mathrm{MALG}_{\mu_1}) \subseteq \ldots$$

We will show next that there is a countable group Γ and a measure preserving action $a_\infty \in A(\Gamma, X_\infty, \mu_\infty)$, which keeps all the $\mathcal{A}_{\infty,n}$ invariant, thus factors to a measure preserving action $\hat{\pi}_{\infty,n}(a_\infty) = a_n \in A(\Gamma, X_n, \mu_n)$, which has moreover the property that $E_{a_n} = E_n$. Then if we put $E_{a_\infty} = E_\infty$, it follows that the factor of E_∞ via $\pi_{\infty,n}$ is exactly E_n and the appropriate diagrams commute.

To construct a_∞, let, for each m, $T_0^m, T_1^m, \dots, T_i^m, \dots$ be in $[E_m]^*$ and generate E_m (using Proposition 17.8). For $n \leq m$, let $T_i^{m,n} = \hat{\pi}_{m,n}(T_i^m)$ and for $n > m$ choose $T_i^{m,n} \in [E_n]^*$ such that $\hat{\pi}_{n+1}(T_i^{m,n+1}) = T_i^{m,n}$ for $n \geq m$ (again using Proposition 17.8). Finally let $T_i^{m,\infty} = (T_i^{m,n})_{n\in\mathbb{N}} \in \mathrm{Aut}(X_\infty, \mu_\infty)$, where $T_i^{m,\infty}((x_n)) = T_i^{m,n}(x_n)$. Note that $T_i^{m,\infty}$ leaves each $\mathcal{A}_{\infty,n}$ invariant and $\hat{\pi}_{\infty,n}(T_i^{m,\infty}) = T_i^{m,n}$.

Let Γ be the free group with infinitely many generators $g_{i,m}$ and let it act in a measure preserving way on (X_∞, μ_∞) to produce a_∞, where $g_{i,m}^{a_\infty} = T_i^{m,\infty}$. If $\hat{\pi}_{\infty,n}(a_\infty) = a_n$, then $g_{i,n}^{a_n} = \hat{\pi}_{\infty,n}(T_i^{n,\infty}) = T_i^{n,n} = T_i^n$, so $E_{a_n} = E_n$ and thus the factor of E_∞ by $\pi_{\infty,n}$ is equal to E_n, which completes the proof.

Although E_∞ is an "upper bound" for the inverse system (E_n), it is not clear how to construct a canonical upper bound, i.e., an inverse limit in the categorical sense for this inverse system.

Next we show that hyperfiniteness is preserved under factoring.

Proposition 17.9. *If E is hyperfinite and F is a factor of E, then F is hyperfinite.*

Proof. Let $\pi\colon (X, \mu) \to (Y, \nu)$ be the factor map and let $a \in A(\Gamma, X, \mu)$ be such that $E_a = E$, $\hat{\pi}(a) = b$ and $E_b = F$. For $y \in Y$, let $X_y = \{x \in X \colon \pi(x) = y\}$ and let μ_y be the probability measure on X_y associated with the measure disintegration of π. Since E is amenable, let $\lambda^n\colon E \to [0,1]$ be Borel functions such that

$$\sum_{x' E x} \lambda^n(x, x') = 1,$$

$$||\lambda^n_x - \lambda^n_u||_1 \to 0, \text{for } xEu, \text{as } n \to \infty$$

(see [KM, Section 9]).

Define now $\rho^n\colon F \to [0,1]$ by

$$\rho^n(y, y') = \int_{X_y} \sum_{xEx', \pi(x')=y'} \lambda^n(x, x')\, d\mu_y(x) \in [0,1].$$

We will show that

$$\sum_{y' F y} \rho^n(y, y') = 1,$$

$$||\rho_y^n - \rho_v^n||_1 \to 0, \text{for } yFv, \text{as } n \to \infty,$$

which implies that F is amenable, thus hyperfinite (see [KM, Section 10]).

The first equality is easy to check, so we verify the second. Fix $\gamma \in \Gamma$ such that $\gamma \cdot y = v$. Then $\gamma \cdot X_y = X_v$ and $\gamma \cdot \mu_y = \mu_v$. Now for each $y'Fy$, we have

$$\rho_y^n(y') = \int_{X_y} \sum_{xEx', \pi(x')=y'} \lambda^n(x, x')\, d\mu_y(x),$$

and

$$\rho_v^n(y') = \int_{X_v} \sum_{xEx', \pi(x')=y'} \lambda^n(x, x')\, d\mu_v(x),$$

so

$$\rho_v^n(y') = \int_{X_y} \sum_{xEx', \pi(x')=y'} \lambda^n(\gamma \cdot x, x')\, d\mu_y(x).$$

It follows that

$$||\rho_y^n - \rho_v^n||_1 \le \int_{X_y} ||\lambda_x^n - \lambda_{\gamma \cdot x}^n||_1\, d\mu_y(x) \to 0,$$

by Lebesgue Dominated Convergence. □

This result can be used, along with an ultraproduct argument, to give a different proof of a strengthening concerning weak containment of actions, due to Robin Tucker-Drob (private communication). We first need a lemma, which extends Proposition 5.7 of [CKT] and Corollary 3.1 of [AE]. Below we let $a \simeq b \iff a \preceq b \,\&\, b \preceq a$ denote the **weak equivalence** of the actions a, b and let $a \sqsubseteq b$ denote that the action a is a factor of the action b.

Lemma 17.10. *Let Γ, Δ be infinite countable groups, $a, b \in A(\Gamma, X, \mu)$, $c \in A(\Delta, X, \mu)$ be such that $a \preceq b$ and $E_b \subseteq E_c$. The there are $d \in A(\Gamma, X, \mu)$, $e \in A(\Delta, X, \mu)$ such that $b \simeq d$, $c \simeq e$, $a \sqsubseteq d$ and $E_d \subseteq E_e$. Similarly replacing $E_b \subseteq E_c$, $E_d \subseteq E_e$ by $E_b = E_c$, $E_d = E_e$, resp.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and consider the ultrapowers $a_\mathcal{U}, b_\mathcal{U}, c_\mathcal{U}$ on the space $(X_\mathcal{U}, \mu_\mathcal{U})$. For each $g \in \Gamma, h \in \Delta$, let $A_{g,h}$ be a Borel set such that for each g, $\bigcup_h A_{g,h} = X$ and $g^b|A_{g,h} = h^c|A_{g,h}$. These can be found as $E_b \subseteq E_c$. Then, using Proposition 16.1, we have that for each g, $\bigcup_h [A_{g,h}]_\mathcal{U} = X_\mathcal{U}$ and $g^{b_\mathcal{U}}|[A_{g,h}]_\mathcal{U} = h^{c_\mathcal{U}}|[A_{g,h}]_\mathcal{U}$. Then as in

[CKT, Sections 4.2 and 5.2] we can find an appropriate countably generated, non-atomic, invariant under $b_{\mathcal{U}}, c_{\mathcal{U}}$, σ-subalgebra $\boldsymbol{B}$ of the measure algebra of $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$, which contains all the sets $[A_{g,h}]_{\mathcal{U}}$ and is such that if $d \in A(\Gamma, X, \mu)$, $e \in A(\Delta, X, \mu)$, resp., are the factors of $b_{\mathcal{U}}, c_{\mathcal{U}}$ corresponding to $\boldsymbol{B}$, then $b \simeq d, c \simeq e$ and $a \sqsubseteq d$. Since $\boldsymbol{B}$ also contains the sets $[A_{g,h}]_{\mathcal{U}}$, it follows that $E_d \subseteq E_e$.

The proof in the case of equality instead of inclusion, as in the last statement of this lemma, is similar. □

Corollary 17.11 (Tucker-Drob). *Let Γ be an infinite countable group and $a, b \in A(\Gamma, X, \mu)$ be such that $a \preceq b$. If E_b is hyperfinite, then E_a is hyperfinite.*

Proof. Apply Lemma 17.10 with $\Delta = \mathbb{Z}$ and use Proposition 17.9. □

Remark 17.12. Standard factors of the ultraproduct $\prod_n F_n/\mathcal{U}$ (which was defined in Chapter 16), can be constructed as in the proof of Theorem 15.10.

17.2 Class-bijective factors

We now consider the following notion that has been considered in the literature, see Feldman-Sutherland-Zimmer [FSZ]). A measure preserving countable Borel equivalence relation F on (Y, ν) is called a **class-bijective factor** of a measure preserving countable Borel equivalence relation E on (X, μ) if there is Borel $\pi\colon (X, \mu) \to (Y, \nu)$ with $\pi_*\mu = \nu$, $\pi\colon E \to F$ a homomorphism (i.e., $xEx' \Rightarrow \pi(x) F \pi(x')$) such that moreover for each E-class $[x]_E$ the map π is a bijection of $[x]_E$ with $[\pi(x)]_F$. In this case we also call the map π **class-bijective**. For example, let E be measure preserving on (X, μ), $\mathcal{A} \subseteq \mathrm{MALG}_\mu$ a σ-subalgebra which generates E, $\pi\colon (X, \mu) \to (Y, \nu)$ the corresponding map, $a \in A(\Gamma, X, \mu)$ with $E_a = E$ leaving $\mathcal{A}$ invariant, $\hat{\pi}(a) = b$ and $F = E_b$. If b is free, then F is a class-bijective factor of E.

Proposition 17.13. *A class-bijective factor is a factor in the sense of Section 17.1.*

Proof. Let $b \in A(\Gamma, Y, \nu)$ be such that $E_b = F$. Define then $a \in A(\Gamma, X, \mu)$ by

$$g^a(x) = x' \iff xEx' \;\&\; g^b(\pi(x)) = \pi(x').$$

Then $\pi(g^a(x)) = g^b(\pi(x))$. Let $\mathcal{A}$ be the σ-subalgebra of MALG_μ corresponding to π. Clearly a preserves $\mathcal{A}$, since $g^a(\pi^{-1}(A)) = \pi^{-1}(g^b(A))$, for any $A \in \mathrm{MALG}_\nu$, and $E_a = E$ while $\hat{\pi}(a) = b$, so F is a factor in the preceding sense. $\square$

Thus a class-bijective factor is a factor π for which $E_\pi = id$. In fact it turns out that the class-bijective factors of a measure preserving countable Borel equivalence relation E on (X, μ) correspond exactly to smooth equivalence relations R_π that commute with E and are orthogonal to E in the sense that $E_\pi = R_\pi \cap E = id$. Indeed, if F on Y is a class-bijective factor of E via π, then E, R_π commute and $E_\pi = id$.

Conversely if E, R_π commute and $E_\pi = id$, define the following relation on Y:

$$xFy \iff \exists x' \exists y' (x' E y' \wedge \pi(x') = x \wedge \pi(y') = y).$$

Then F is a equivalence relation on Y (transitivity follows from the commutativity of E, R_π). It is clearly analytic. It is also coanalytic, since, by the commutativity of E, R_π, we also have:

$$xFy \iff \forall x' (\pi(x') = x \implies \exists y' (x' E y' \wedge \pi(y') = y)).$$

Thus F is Borel. Moreover the map π is bijective from $[x]_E$ to $[\pi(x)]_F$ (using that $E_\pi = id$), and so, in particular, F is a countable equivalence relation. Finally, it is easy to verify that F is measure preserving.

Since $[E_\pi] = [E]_{\mathcal{A}}$ we also immediately have:

Proposition 17.14. *Assume that E on (X, μ) is generated by the σ-subalgebra $\mathcal{A} \subseteq \mathrm{MALG}$ with corresponding map $\pi \colon (X, \mu) \to (Y, \nu)$ and factor F. Then π is class-bijective iff $[E]_{\mathcal{A}}$ is trivial, i.e., $\hat{\pi}$ is an isomorphism of $[E]^{\mathcal{A}}$ with $[F]$.*

We will next characterize which factors are class-bijective. Below for each $T \in \mathrm{Aut}(X, \mu)$, we let as usual $\mathrm{supp}(T) = \{x \colon T(x) \neq x\}$.

Proposition 17.15. *Assume that E on (X, μ) is generated by the σ-subalgebra $\mathcal{A} \subseteq \mathrm{MALG}$ with corresponding map $\pi \colon (X, \mu) \to (Y, \nu)$ and factor F. Then π is class-bijective iff for each $T \in [E]^{\mathcal{A}}$, $\mathrm{supp}(T) = \pi^{-1}(\mathrm{supp}(\hat{\pi}(T)))$.*

Proof. First note that for any $T \in [E]^{\mathcal{A}}$, we have

$$\hat{\pi}(T)(\pi(x)) \neq \pi(x) \iff \pi(T(x)) \neq \pi(x),$$

so

$$\pi^{-1}(\mathrm{supp}(\hat{\pi}(T)) \subseteq \mathrm{supp}(T).$$

Now assume that π is class-bijective. Let $T \in [E]^{\mathcal{A}}$ and $T(x) \neq x$. Then as π is 1-1 on $[x]_E$, $\hat{\pi}(T)(\pi(x)) \neq \pi(x)$, so $\pi^{-1}(\mathrm{supp}(\hat{\pi}(T)) \supseteq \mathrm{supp}(T)$.

Conversely, assume that $\pi^{-1}(\mathrm{supp}(\hat{\pi}(T)) \supseteq \mathrm{supp}(T)$ and let $x \neq x' \in [x]_E$. Then for some $T \in [E]^{\mathcal{A}}$ we have $T(x) = x'$, so $x \in \mathrm{supp}(T)$, thus $\pi(x) \in \mathrm{supp}(\hat{\pi}(T))$, so $\pi(x') = \pi(T(x)) = \hat{\pi}(T)(\pi(x)) \neq \pi(x)$, i.e., π is 1-1 on $[x]_E$. □

From Proposition 17.15, and using its notation, we see that if π is class-bijective, then for $T \in [E]^{\mathcal{A}}$ we have that $\mathrm{supp}(T) \in \mathcal{A}$. Conversely this last condition almost characterizes class-bijective factors. Recall that π is class-bijective iff $\mathrm{card}([x]_{E_\pi}) = 1$, for all x.

Proposition 17.16. *Assume that E on (X, μ) is generated by the σ-subalgebra $\mathcal{A} \subseteq \mathrm{MALG}$ with corresponding factor map $\pi\colon (X, \mu) \to (Y, \nu)$. If for every $T \in [E]^{\mathcal{A}},\ \mathrm{supp}(T) \in \mathcal{A}$, then $\mathrm{card}([x]_{E_\pi}) \leq 2$, for all x.*

Proof. Assume the conclusion fails, towards a contradiction. Let $A_\infty = \{x\colon \mathrm{card}([x]_{E_\pi}) = \infty\}$ and $A_{\geq 3} = \{x\colon \infty > \mathrm{card}([x]_{E_\pi}) \geq 3\}$. Then one of these two sets has positive measure.

Case 1. $\mu(A_\infty) > 0$. Let then $B \subseteq A_\infty$ be a Borel set such that both B and $A_\infty \setminus B$ meet every $E_\pi|A_\infty$ class. The by [K, 4.10] there is $T_0 \in [E_\pi|A_\infty]$ with $\mathrm{supp}(T_0) = B$. Extend T_0 to $T \in [E_\pi] = [E]_{\mathcal{A}}$ by letting $T(x) = x$ for $x \notin A_\infty$. Then $\mathrm{supp}(T) = B$ but B is not E_π-invariant, so $B \notin \mathcal{A}$, a contradiction.

Case 2. $\mu(A_{\geq 3}) > 0$. Let then $C \subseteq A_{\geq 3}$ be a Borel selector for $E_\pi|A_{\geq 3}$. Then $\mu(C) > 0$. Define $T_1 \in [E_\pi|A_{\geq 3}]$ so that $x \in C \implies (T_1(x) \neq x \wedge T_1^2(x) = x)$ and $x \notin (C \cup T_1(C)) \implies T_1(x) = x$. Extend T_1 to $T \in [E_\pi]$ by letting $T(x) = x$ if $x \notin A_{\geq 3}$. Since $\mathrm{supp}(T) = C \cup T(C)$ is not E_π-invariant, so not in $\mathcal{A}$, we again have a contradiction. □

That the conclusion of Proposition 17.16 cannot be strengthened to π being class-bijective can be seen from the following example. Let E on (Y, ν) be given, let $X = Y \times \{0, 1\}$, with the product measure μ, and let $(x, i)E(y, j) \iff xFy$. Then for $\pi\colon X \to Y$ the projection function, the hypothesis of Proposition 17.16 is satisfied but π is clearly not class-bijective.

Class-bijective factors can be also characterized, in the ergodic case, in terms of skew products. Let F be a measure preserving equivalence relation on (Y, ν). Let (Z, σ) be a standard, not necessarily non-atomic, measure space and let $\alpha\colon F \to \mathrm{Aut}(Z, \sigma)$ be a Borel cocycle, i.e., $\alpha(x, z) =$

$\alpha(y,z)\alpha(x,y)$ for $xFyFz$ (in an F-invariant set of measure 1). Let $X = Y \times Z, \mu = \nu \times \sigma$ and define the **skew product** equivalence relation E on X, in symbols

$$E = F \times_\alpha (Z, \sigma),$$

by

$$(x,z)E(y,w) \iff xFy \;\&\; \alpha(x,y)(z) = w.$$

Let $p\colon X \to Y$ be the projection map $p(y,z) = y$. Let $a \in A(\Gamma, Y, \nu)$ be such that $E_a = F$. Let also $\alpha^*(g,y) = \alpha(y, g^a(y))$. Then if $b = a \times_{\alpha^*} (Z, \sigma)$ is the skew product action (see [K, Section 10, **(E)**]), we have $E_b = E$ and since $\hat{p}(b) = a$, it follows that F is the factor of E corresponding to p. Moreover it is easy to see that it is class-bijective.

Conversely, the proof of Rokhlin's Skew Product Theorem (see Glasner [Gl], 3.18) shows that if F on (Y,ν) is a class-bijective factor of an ergodic E on a space (X,μ) via $\pi\colon (X,\mu) \to (Y,\nu)$, then there is a standard, not necessarily non-atomic, space (Z,σ), a Borel cocycle $\alpha\colon F \to \mathrm{Aut}(Z,\sigma)$ and an isomorphism $\varphi\colon (X,\mu) \to (Y \times Z, \nu \times \sigma)$ of E with $F \times_\alpha (Z,\sigma)$ such that $p \circ \varphi = \pi$.

If F on (Y,ν) is a (class-bijective) factor of E on (X,μ) via π, we say that E is a (class-bijective) **extension** of E via π. Given two such extensions E, E' of F on $(X,\mu), (X',\mu')$ via π, π', we say that they are isomorphic if there is an isomorphism $\varphi\colon (X,\mu) \to (X',\mu')$ of E with E' with $\pi' \circ \varphi = \pi$. Thus we have shown the following:

Theorem 17.17. *Let F be an ergodic measure preserving equivalence relation on (Y,ν). Let E be an ergodic extension of F on (X,μ) via $\pi\colon (X,\mu) \to (Y,\nu)$. Then the following are equivalent:*

(i) E, π is a class-bijective extension of F.

(ii) E, π is isomorphic to a skew product extension of F.

Concerning the question of inverse limits for systems $((X_n,\mu_n),\pi_n,E_n)$ we note that if we restrict ourselves to the category of class-bijective factors, i.e., if in this system every factor is class-bijective, then it is easy to see that there is indeed a canonical inverse limit $E_\infty = \varprojlim_n E_n$ on (X_∞,μ_∞), given by

$$(x_n)E_\infty(y_n) \iff \forall n(x_n E_n y_n).$$

This follows from the "unique lifting property" given in Proposition 17.14, which implies that if $a_0 \in A(\Gamma, X_0, \mu_0)$ is such that $E_{a_0} = E_0$, then there are unique $a_n \in A(\Gamma, X_n, \mu_n)$ with $\hat{\pi}_{n,m}(a_n) = a_m$ for $n \geq m$ and $a_\infty \in A(\Gamma, X_\infty, \mu_\infty)$ with $\hat{\pi}_{\infty,n}(a_\infty) = a_n$ such that $E_{a_n} = E_n, E_{a_\infty} = E_\infty$.

The following is an interesting open problem:

Problem 17.18. *If E is treeable and F is a class-bijective factor of E, is F treeable?*

Note that a positive answer implies that every countable treeable group Γ is strongly treeable. (Recall that a countable group Γ is **treeable** if there is *some* free $a \in A(\Gamma, X, \mu)$ with E_a treeable, while it is **strongly treeable** if this holds for *every* free $a \in A(\Gamma, X, \mu)$.) Indeed let $a \in A(\Gamma, X, \mu)$ be free with E_a treeable and consider any free $b \in A(\Gamma, Y, \nu)$. Let $a \times b$ be the product of a, b. Then $E_{a\times b}$ is a class-bijective extension of E_a, so it is treeable. Also E_b is a class-bijective factor of $E_{a\times b}$, so, if the answer to Problem 17.18 is positive, E_b is treeable.

17.3 Other notions of factors

In the preceding we have considered two categories whose objects are triples (X, μ, E), with E a countable measure preserving Borel equivalence relation on (X, μ).

(1) In the first category, the morphisms $\pi\colon (X, \mu, E) \to (Y, \nu, F)$ are measure preserving Borel maps $\pi\colon (X, \mu) \to (Y, \nu)$ with $\pi\colon E \to F$ a **class-bijective homomorphism**, i.e., for each $x \in X$, π is a bijection of $[x]_E$ with $[\pi(x)]_E$. (The notation $\pi\colon E \to F$, which more accurately should be written as $\pi \times \pi\colon E \to F$, indicates that π is a homomorphism of E into F.)

(2) In the second category, the morphisms $\pi\colon (X, \mu, E) \to (Y, \nu, F)$ are measure preserving Borel maps $\pi\colon (X, \mu) \to (Y, \nu)$ such that if $\mathcal{A} \subseteq \mathrm{MALG}_\mu$ is the σ-algebra associated to π, then $[E]^{\mathcal{A}}$ generates E and $\hat{\pi}([E]^{\mathcal{A}})$ generates F (or equivalently there is Borel action a of a countable group Γ preserving $\mathcal{A}$, such that $E_a = E$ and $E_{\hat{\pi}(a)} = F$).

Robin Tucker-Drob (unpublished) considered the following two additional categories with the same objects (X, μ, E).

(3) In the third category, the morphisms $\pi\colon (X, \mu, E) \to (Y, \nu, F)$ are measure preserving Borel maps $\pi\colon (X, \mu) \to (Y, \nu)$ with $\pi\colon E \to F$ a **class-surjective homomorphism**, i.e., for each $x \in X$, π is a surjection of $[x]_E$

with $[\pi(x)]_E$. Note that a class-surjective homomorphism is a morphism in the sense of the second category (i.e., that of Section 17.1) iff the homomorphism $\hat{\pi}\colon [E]^{\mathcal{A}} \to [F]$ is surjective. One direction follows from Proposition 17.1. For the other direction, recall that $xR_\pi y \iff \pi(x) = \pi(y)$ and $E_\pi = E \cap R_\pi$. Let $S_0, S_1, \dots \in [F]$ generate F and let $T_0, T_1, \dots \in [E]^{\mathcal{A}}$ be such that $\hat{\pi}(T_i) = S_i$. Let also $U_0, U_1, \dots \in [E_\pi]$ generate E_π. Then $T_0, T_1, \dots, U_0, U_1, \dots$ generate E and of course $\hat{\pi}(T_0), \hat{\pi}(T_1), \dots$ generate F.

(4) Finally, in the fourth category, the morphisms

$$\pi\colon (X, \mu, E) \to (Y, \nu, F)$$

are measure preserving Borel maps $\pi\colon (X, \mu) \to (Y, \nu)$ with $\pi\colon E \to F$ a **surjective homomorphism** (i.e., $(\pi \times \pi)(E) = F$).

Note that the categories above have the same objects but increasingly more general morphisms.

17.4 An application to soficity

We start with the following proposition.

Proposition 17.19. *Let $\pi\colon E \to F$ be a class-surjective homomorphism. Assume that F is treeable. Then the following are equivalent:*

(i) F is a factor of E via π.

(ii) There is Borel $E' \subseteq E$ such that F is a class-bijective factor of E' via π and $E = E' \vee E_\pi$.

Proof. (ii) $\implies$ (i): Let $T_0, T_1, \dots \in [E_\pi]$ generate E_π. If $\mathcal{A}$ is the σ-algebra corresponding to π, clearly $T_n \in [E]_{\mathcal{A}}$. Let also $T'_0, T'_1, \dots \in [E']^{\mathcal{A}} \subseteq [E]^{\mathcal{A}}$ generate E' with $\hat{\pi}(T'_0), \hat{\pi}(T'_1), \dots$ generating F. Then $T_0, T_1, \dots, T'_0, T'_1, \dots$ generate E. Note that we have only used that F is a factor of E' here.

(i) $\implies$ (ii): Fix a Borel treeing of F (i.e, a Borel acyclic graph whose connected components are the F-classes). Using a Borel edge coloring of this treeing with countably many colors (see [KST, Proposition 4.10]), we can find a (finite or infinite) sequence $S_0, S_1, \dots$ of Borel involutions generating this treeing, so that if $m \neq n$ and $S_m(x) = y$ with $x \neq y$, then $S_n(x) \neq y$.

By Proposition 17.3, let $T_n \in [E]^{\mathcal{A}}$ be an involution such that $\hat{\pi}(T_n) = S_n$. We can clearly assume that T_n is chosen so that $T_n(x) = x$ if $S_n(\pi(x)) =$

$\pi(x)$. Let E' be the equivalence relation generated by the T_n, so that $E' \subseteq E$. We will show that π is a class-bijective homomorphism of E' to F.

Clearly π is a class-surjective homomorphism of E' to F. To check that it is class-bijective, let $x'E'y', x' \neq y'$ but, towards a contradiction, $\pi(x') = \pi(y')$. Let n be least such that we can find $T_{i_1}, T_{i_2}, \ldots, T_{i_n}$ with $T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}(x') = y'$ and therefore $S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n}(\pi(x')) = \pi(y')$, contradicting the acyclicity of the treeing.

Finally $E = E' \vee E_\pi$ follows from Proposition 17.2. $\square$

Definition 17.20. *Let F be a countable measure preserving Borel equivalence relation on (Y, ν). We say that F is* **unfoldable** *if for any E on (X, μ) which factors to F via $\pi\colon (X, \mu) \to (Y, \nu)$, there is $E' \subseteq E$ such that F is a class-bijective factor of E' via π.*

Thus every treeable equivalence relation is unfoldable. For the next result recall the notion of a **sofic** equivalence relation introduced in [EL]. See also [CKT, Definition 10.1] for an alternative description due to Ozawa that we will use below.

Proposition 17.21. *Every unfoldable equivalence relation is sofic.*

Proof. First notice that every F is a factor of an E that is given by a free action of $\mathbb{F}_\infty$. Indeed let a be an action of $\mathbb{F}_\infty$ such that $F = E_a$ and let b be a free action of $\mathbb{F}_\infty$. Then take $E = E_{a \times b}$. Since E is given by a free action of the group $\mathbb{F}_\infty$, E is sofic (this follows from the fact that $\mathbb{F}_\infty$ has property MD – see the first two paragraphs of [CKT, Section 10.3])

Assume now that F is unfoldable and let $E' \subseteq E$ be such that F is a class-bijective factor of E'. Then clearly $[[E']] \subseteq [[E]]$ and, since F is a class-bijective factor of E', there is a canonical embedding of $[[F]]$ into $[[E']]$ and thus into $[[E]]$, so, by the definition of soficity, F is sofic. $\square$

The combination of Proposition 17.19, Proposition 17.21 gives then a new proof of the following result of Elek-Lippner (another proof is also given in [CKT, Section 10.3]).

Corollary 17.22 (Elek-Lippner, [EL])**.** *Every treeable equivalence relation is sofic.*

17.5 Relative hyperfiniteness

We consider here the following question:

Suppose E is hyperfinite and generated by a non-atomic σ-subalgebra $\mathcal{A}$, i.e., E is generated by a countable group of transformations that are $\mathcal{A}$-measurable (i.e., preserve $\mathcal{A}$). Can we find a single $\mathcal{A}$-measurable transformation that generates E, i.e., is E hyperfinite relative to $\mathcal{A}$?

The answer is in general negative as the following example shows: Consider $(2^{\mathbb{N}}, \mu)$, where μ is the usual product measure, and the equivalence relation E_0 of eventual equality. Let $E = E_0 \times E_0$ be the product equivalence relation, let $\pi\colon 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the first projection and $\mathcal{A}$ the corresponding σ-algebra. Clearly E is generated by $\mathcal{A}$. Suppose, towards a contradiction, that there is a T which preserves $\mathcal{A}$ and generates E. Then T sends vertical lines (i.e., sets of the form $\pi^{-1}(x)$) to vertical lines and $\hat{\pi}(T)$ generates E_0, so it is aperiodic, thus T fixes no vertical line. But in every vertical line there are distinct E-inequivalent elements, so T cannot generate E.

The following result provides the next possible answer.

Theorem 17.23. *Let E be hyperfinite and generated by a non-atomic σ-subalgebra $\mathcal{A}$. Then*

(i) There are $T_1, T_2 \in [E]^{\mathcal{A}}$ that generate E.

(ii) If E is ergodic, then there is $T \in [E]^{\mathcal{A}}$ that generates E iff the factor corresponding to $\mathcal{A}$ is class-bijective.

Proof. (i) Let $\pi\colon X \to Y$ be the map associated to $\mathcal{A}$ and F the corresponding factor equivalence relation on Y. Then, by Proposition 17.9, F is hyperfinite. Say $F = E_S$, where $S \in [F]$. Let $T_1 \in [E]^{\mathcal{A}}$ be such that $\hat{\pi}(T_1) = S$ and let $E' = E_{T_1}$. Then by Proposition 17.2, $E = E' \vee E_\pi$. Clearly E_π is generated by some $T_2 \in [E]^{\mathcal{A}}$, so E is generated by T_1, T_2.

(ii) If the factor corresponding to $\mathcal{A}$ is class-bijective, then, since the factor equivalence relation F is hyperfinite, clearly there is $T \in [E]^{\mathcal{A}}$ that generates E. Conversely, assume that there is $T \in [E]^{\mathcal{A}}$ that generates E and, towards a contradiction, that $E_\pi \neq id$. Then for a positive measure set of x, there is $n \neq 0, n \in \mathbb{Z}$ such that $\pi(T^n(x)) = \pi(x)$. Now $\hat{\pi}(T) = S$ generates F and $S^n(\pi(x)) = \pi(x)$, therefore $[\pi(x)]_F$ is finite. But F is ergodic, so aperiodic, a contradiction. □

Ben Miller raised the following questions:

Problem 17.24. *i) Let E be a hyperfinite equivalence related generated by a non-atomic σ-subalgebra $\mathcal{A}$. Is there is an increasing sequence $E_0 \subseteq E_1 \subseteq \dots$ of finite equivalence relations which are generated by $\mathcal{A}$ with $E = \bigcup_n E_n$?*

ii) What if we assume the stronger hypothesis that $E = E_T$, for some $T \in [E]^{\mathcal{A}}$?

We have the following result which provides a weaker version of a positive answer to part i) of Problem 17.24 and a positive answer to part ii)..

Proposition 17.25. *i) Let E be hyperfinite and generated by a non-atomic σ-subalgebra $\mathcal{A}$. Then there is an increasing sequence $E_0 \subseteq E_1 \subseteq \dots$ of equivalence relations, which are generated by $\mathcal{A}$, with $E = \bigcup_n E_n$ and for each n an increasing sequence $E_{n,0} \subseteq E_{n,1} \subseteq \dots$ of finite equivalence relations which are generated by $\mathcal{A}$ such that $E_n = \bigcup_m E_{n,m}$.*

In particular, E is the limit (in the topology of $S(E)$) of a sequence of finite subequivalence relations which are generated by $\mathcal{A}$.

ii) If moreover $E = E_T$, for some $T \in [E]^{\mathcal{A}}$, then then there is an increasing sequence $E_0 \subseteq E_1 \subseteq \dots$ of finite equivalence relations, which are generated by $\mathcal{A}$, with $E = \bigcup_n E_n$.

Proof. i) Consider the factor map π associated with $\mathcal{A}$ and the factor equivalence relation F. Then F is hyperfinite, by Proposition 17.9, so we can write it as $F = \bigcup_n F_n$, with $F_0 \subseteq F_1 \subseteq \dots$ finite equivalence relations. Let also, by Proposition 17.19, $E' \subseteq$ be such that F is a class-bijective factor of E' via π and $E = E' \vee E_\pi$. Let $xF'_n y \iff xE'y \;\&\; \pi(x)F_n\pi(y)$ and put $E_n = F'_n \vee E_\pi$. Clearly E_n is generated by $\mathcal{A}$, increasing, and $\bigcup_n E_n = E$.

Fix now n in order to define $E_{n,m}$. Let B be a Borel selector for F_n, i.e., a Borel set meeting every F_n-class in exactly one point. For each F_n-class C let y_C be the point of B in C. Write also $E_\pi = \bigcup_m E_{\pi,m}$, with $E_{\pi,m}$ finite and increasing. Clearly each $E_{\pi,m}$ is generated by $\mathcal{A}$. Define now $E_{n,m}$ as follows:

Given x with $\pi(x) = y_C$ and any $y \in C$, there is a unique point $\theta_y(x)$ such that $\pi(\theta_y(x)) = y$ and $xF'_n\theta_y(x)$. Define now the equivalence relation $E_{\pi,n,m}$ by letting $zE_{\pi,n,m}w$ iff $zE_\pi w$ and if C is the F_n-class of $\pi(z) = \pi(w)(= y)$, then there are z', w' with $z'E_{\pi,m}w'$ and $\theta_y(z') = z, \theta_y(w') = w$.

Clearly $E_{\pi,n,m}$ is finite and, since it is contained in E_π, it is generated by $\mathcal{A}$. Finally let $E_{n,m} = E_{\pi,n,m} \vee F'_n$. This works.

ii) Let π, F be as in i) and let $S = \hat{\pi}(T)$. Denote by P the Borel set which is the union of the set of finite F-classes and let Q be the complement of P. Let $A = \pi^{-1}(P), B = \pi^{-1}(Q)$. These are both in $\mathcal{A}$. Clearly $F|Q$ is a class-bijective factor of $E|B$, so since $F|Q$ is the union an increasing sequence of finite equivalence equivalence relations, $E|B$ is the union an increasing sequence of finite equivalence equivalence relations, which are generated by $\mathcal{A}|B$. It is thus enough to show that $E|A$ is the union an increasing sequence of finite equivalence equivalence relations, which are generated by $\mathcal{A}|A$. This can be done exactly as in the construction of the $E_{n,m}$ from F_n in part i). $\square$

One can also ask if a kind of converse of the Problem 17.24, ii) is true: If there is an increasing sequence $E_0 \subseteq E_1 \subseteq \dots$ of finite equivalence relations, which are generated by $\mathcal{A}$, with $E = \bigcup_n E_n$, is there $T \in [E]^{\mathcal{A}}$ such that $E = E_T$? The example given before Theorem 17.23 shows that this fails in general.

Remark 17.26. Let E be a measure preserving countable Borel equivalence relation on (X, μ). Then of course the following are equivalent:

a) $E = E_T$ for some $T \in \mathrm{Aut}(X, \mu)$,

b) E is the union of an increasing sequence of finite Borel equivalence relations.

The preceding show that when relativized to a σ-subalgebra $\mathcal{A}$, a) implies b) but not vice versa.

17.6 Relative cost

Let E be a measure preserving countable Borel equivalence relation on (X, μ) and let $\mathcal{A}$ be a non-atomic σ-subalgebra of MALG such that E is generated by $\mathcal{A}$. Let $\pi\colon X \to Y$ be the associated to $\mathcal{A}$ factor map and F the factor equivalence relation. Define the **relative to $\mathcal{A}$ full pseudogroup** of E, in symbols $[[E]]^{\mathcal{A}}$, as the set of all partial Borel bijections $\theta \in [[E]]$, $\theta\colon A \to B$, such that $A, B \in \mathcal{A}$, and for any $A' \subseteq A, B' \subseteq B, A', B' \in \mathcal{A}$, we have $\theta(A'), \theta^{-1}(B') \in \mathcal{A}$. If $\theta \in [[E]]^{\mathcal{A}}, \theta\colon A \to B$, and $A = \pi^{-1}(C), B = \pi^{-1}(D)$, then, as in Section 17.1, we have an element $\hat{\pi}(\theta) \in [[F]]$ such that $\hat{\pi}(\theta)\colon C \to D$ and $\hat{\pi}(\theta)(\pi(x)) = \pi(\theta(x))$, for $x \in A$. Moreover, as in the proof of Proposition 17.1, the map $\hat{\pi}\colon [[E]]^{\mathcal{A}} \to [[F]]$ is surjective and preserves composition.

Next define the **cost of E relative to $\mathcal{A}$** by

$$C^{\mathcal{A}}(E) = \inf\{\sum_{i\in I} \mu(A_i)\colon\ \theta_i\colon A_i \to B_i \in [[E]]^{\mathcal{A}}, (\theta_i)_{i\in I} \text{ generates } E\}$$

(where I varies over countable index sets).

Clearly $C^{\mathcal{A}}(E) \geq C(E)$. Also notice that if $(\theta_i)_{i\in I}$ generates E, then $(\hat{\pi}(\theta_i))_{i\in I}$ generates F, therefore $C^{\mathcal{A}}(E) \geq C(F)$.

Below we say that an equivalence relation E on (X, μ) is **finitely generated** if it is of the form $E = E_{T_1,\dots,T_n}$, for some $T_1, \dots, T_n \in \mathrm{Aut}(X, \mu)$.

Theorem 17.27. *Let E be a measure preserving countable Borel equivalence relation on (X, μ) and let $\mathcal{A}$ be a non-atomic σ-subalgebra of* MALG *that generates E. Let $\pi\colon (X, \mu) \to (Y, \nu)$ be the associated to $\mathcal{A}$ factor map and F the factor equivalence relation. If F is aperiodic (e.g., if E is ergodic) and E_π is finitely generated, then $C^{\mathcal{A}}(E) = C(F)$.*

Proof. We have already seen that $C^{\mathcal{A}}(E) \geq C(F)$. If $C(F) = \infty$, then $C^{\mathcal{A}}(E) = C(F) = \infty$. So we can assume that $C(F) < \infty$. We will show then that $C^{\mathcal{A}}(E) \leq C(F)$.

Let $\epsilon > 0$ and find a graphing $(\eta_i)_{i\in I}$ of F (where $\eta_i \in [[F]]$) such that $\sum_i \nu(\mathrm{dom}(\eta_i)) < C(F) + \epsilon$. Let $\theta_i \in [[E]]^{\mathcal{A}}$ be such that $\hat{\pi}(\theta_i) = \eta_i$. Then $\sum_i \mu(\mathrm{dom}(\theta_i)) < C(F) + \epsilon$.

Let $E' \subseteq E$ be the equivalence relation generated by $(\theta_i)_{i\in I}$. We claim that $E = E' \vee E_\pi$. Indeed let xEy. Then $\pi(x)F\pi(y)$, so there are $i_1, \dots, i_n$ such that $\eta_{i_1}^{\pm 1} \circ \cdots \circ \eta_{i_n}^{\pm 1}(\pi(x)) = \pi(\theta_{i_1}^{\pm 1} \circ \cdots \circ \theta_{i_n}^{\pm 1}(x)) = \pi(y)$, so $\theta_{i_1}^{\pm 1} \circ \cdots \circ \theta_{i_n}^{\pm 1}(x) E_\pi y$.

Since F is aperiodic, fix a Borel complete section B of F with $\nu(B) < \epsilon$, so that if $A = \pi^{-1}(B)$, then $\mu(A) < \epsilon$. Let

$$E_A = E_\pi | A \sqcup \mathrm{id}|(X \setminus A).$$

Since A is R_π-invariant, so E_π-invariant, E_A is generated by the maps $T_1|A, \dots, T_n|A$, which belong to $[[E_A]]^{\mathcal{A}} \subseteq [[E]]^{\mathcal{A}}$ (note that $T_i(C) = C$, for any $C \in \mathcal{A}$).

We next claim that $E = E' \vee E_A$. Indeed it is enough to show that $E_\pi \subseteq E' \vee E_A$. Let $xE_\pi y$. Since B is a complete section of F, there are $i_1, \dots, i_n$ such that $\eta_{i_1}^{\pm 1} \circ \cdots \circ \eta_{i_n}^{\pm 1}(\pi(x)) \in B$, so $\theta_{i_1}^{\pm 1} \circ \cdots \circ \theta_{i_n}^{\pm 1}(x) \in A$. Also $\theta_{i_1}^{\pm 1} \circ \cdots \circ \theta_{i_n}^{\pm 1}(x) E_\pi \theta_{i_1}^{\pm 1} \circ \cdots \circ \theta_{i_n}^{\pm 1}(y) \in A$, thus $xE'\theta_{i_1}^{\pm 1} \circ \cdots \circ \theta_{i_n}^{\pm 1}(x) E_A \theta_{i_1}^{\pm 1} \circ \cdots \circ \theta_{i_n}^{\pm 1}(y) E' y$.

We now have that E is generated by the $\theta_i, i \in I$, and $T_1|A, \dots, T_n|A$ which are all in $[[E]]^{\mathcal{A}}$ and the sum of the measure of their domains is $< C(F) + (n+1)\epsilon$, thus, letting $\epsilon \to 0$, we have that $C^{\mathcal{A}}(E) \le C(F)$. $\square$

Corollary 17.28. *Let $E, \mathcal{A}, \pi, F$ be as in Theorem 17.27. Then if E_π is hyperfinite, $C^{\mathcal{A}}(E) = C(F)$. In particular, if E is ergodic hyperfinite, then $C^{\mathcal{A}}(E) = 1$.*

Although for E ergodic hyperfinite there might not be a single automorphism $T \in [E]^{\mathcal{A}}$ that generates E (see Theorem 17.23), Corollary 17.28 shows that $C^{\mathcal{A}}(E)$ is still equal to 1.

It turns out that the hypothesis that E_π is finitely generated is needed in Theorem 17.27. This can be seen from the following example:

Let N be a non-trivial, normal subgroup of $\mathbb{F}_2$ of infinite index, so that N is a free group of infinite rank. Let $\Gamma = \mathbb{F}_2/N$. Let b' be a free action in $A(\Gamma, Y, \nu)$ and then let $b \in A(\mathbb{F}_2, Y, \nu)$ be the (non-free) action of $\mathbb{F}_2$ induced by b' and the surjective homomorphism of $\mathbb{F}_2$ onto Γ. Then for each $y \in Y$ the stabilizer of y in the action b is equal to N. Let now c be a free action in $A(\mathbb{F}_2, Z, \eta)$ and let $a = b \times c$, which is a free action of $\mathbb{F}_2$ on $(X = Y \times Z, \mu = \nu \times \eta)$. Let $\pi(y, z) = y$ and let $\mathcal{A}$ be the associated σ-subalgebra. Letting $E = E_a, F = E_b$, all the conditions of Theorem 17.27 are satisfied, except for E_π being finitely generated. Indeed notice that E_π is generated by the free action of N on X, so has infinite cost, thus cannot be finitely generated. We will now see that the conclusion of Theorem 17.27 fails. First notice that $2 = C(E) \le C^{\mathcal{A}}(E) \le 2$, since the two generators of $\mathbb{F}_2$ (acting on X), say T_1, T_2, are in $[E]^{\mathcal{A}}$. Thus if the conclusion of Theorem 17.27 was true, we would have $C(F) = 2$. Consider then the graphing of F given by $\hat{\pi}(T_1), \hat{\pi}(T_2)$. It has cost 2, so it attains the cost of F, thus it is a treeing (Gaboriau, see, e.g., [KM, 19.1]), which implies that the action of $\mathbb{F}_2$ on Y is free, a contradiction.

We next consider the question of when the infimum in the definition of $C^{\mathcal{A}}(E)$ is attained.

Proposition 17.29. *Let $E, \mathcal{A}, \pi, F$ be as in Theorem 17.27. Then*

i) If π is class-bijective and F is treeable, the infimum in the definition of $C^{\mathcal{A}}(E)$ is attained.

ii) Conversely, if the infimum in the definition of $C^{\mathcal{A}}(E)$ is attained and F has finite cost, F is treeable and π is class-bijective.

Proof. i) Note that if (η_i) is a treeing of F, which therefore attains the cost of F (Gaboriau, see, e.g., [KM, 27.10]), and $\hat{\pi}(\theta_i) = \eta_i$, with $\theta_i \in [[E]]^{\mathcal{A}}$, then (θ_i) attains $C^{\mathcal{A}}(E) = C(F)$.

ii) Assume now (θ_i) attains $C^{\mathcal{A}}(E) = C(F)$. Then if $\hat{\pi}(\theta_i) = \eta_i$, (η_i) generates F and attains the cost of F, so it is a treeing of F, since F has finite cost (Gaboriau, see, e.g., [KM, 19.1]). If now $E_\pi \neq id$, towards a contradiction, let $xE_\pi y, x \neq y$. Then there are $i_1, \dots, i_n$ such that $\theta_{i_1}^{\pm 1} \circ \dots \circ \theta_{i_n}^{\pm 1}(x) = y$, thus $\eta_{i_1}^{\pm 1} \circ \dots \circ \eta_{i_n}^{\pm 1}(\pi(x)) = \pi(x) = \pi(y)$, a contradiction. □

In particular, using also Proposition 17.9, if E is ergodic hyperfinite, then the infimum in the definition of $C^{\mathcal{A}}(E) = 1$ is attained iff π is class-bijective.

Let E be a measure preserving countable Borel equivalence relation. We define the **cost spectrum** of E, in symbols $CSp(E)$, as the set of all $C^{\mathcal{A}}(E)$, where $\mathcal{A}$ varies over all the non-atomic σ-subalgebras of MALG such that E is generated by $\mathcal{A}$. (Thus $CSp(E) \subseteq [C(E), \infty]$.) Clearly the cost spectrum is an invariant of isomorphism among equivalence relations. It might therefore be interesting to study its structure.

For example, if E is ergodic hyperfinite, then $CSp(E) = \{1\}$. Is it true that if E is ergodic, non-hyperfinite but has cost 1, then $CSp(E) \neq \{1\}$? If in fact for every ergodic, non-hyperfinite E of cost 1, one has an $\mathcal{A}, \pi$ such that actually E_π is finitely generated and $C^{\mathcal{A}}(E) > 1$, then it follows that for every ergodic, non-hyperfinite E there is a subequivalence relation induced by a free action of $\mathbb{F}_2$ (which answers positively [KM, 28.14]). Indeed if that is the case, every ergodic, non-hyperfinite E would have a factor F of cost > 1, so that by [KM, 28.8] it would have a subequivalence relation induced by a free action of $\mathbb{F}_2$, which then could be lifted to such an action of $\mathbb{F}_2$ whose corresponding equivalence relation is included in E.

It is actually easy, using Theorem 17.27, to construct examples of ergodic, non-hyperfinite E of cost 1, whose cost spectrum contains any finite set of reals > 1. Given $1 < c_1 < \dots < c_n$, simply take ergodic, finitely generated equivalence relations $E_1, \dots, E_n$ with $C(E_i) = c_i$ (Gaboriau, see, e.g., [KM, page 125, line 3]) and let $E = E_1 \times \dots \times E_n$. Then E has cost 1 (Gaboriau, see, e.g., [KM, 24.9]) but, using Theorem 17.27 and considering the factors corresponding to the projection functions, we see that $CSp(E) \supseteq \{c_1, \dots, c_n\}$.

17.7 Topological rank of relative full groups

Recall that a **topological generator** of a topological group Γ is a subset Γ_0 of Γ such that the subgroup generated by Γ_0 is dense in Γ. The **topological rank** of Γ, denoted by $t(\Gamma)$, is the smallest cardinality of a topological generator of Γ. Thus if Γ is Polish, then $t(\Gamma) \leq \aleph_0$. It is easy to see that if Γ is a Polish group, $N \lhd \Gamma$ a closed normal subgroup and $H = \Gamma/N$, then $t(\Gamma) \leq t(N) + t(H)$. Indeed, if N_0 is a topological generator of N and H_0 a topological generator of H, then choose for each coset in H_0 a representative and let $\hat{H}_0 \subseteq \Gamma$ consist of these representatives. Then $N_0 \cup \hat{H}_0$ is a topological generator for Γ.

Let now E be a countable measure preserving Borel equivalence relation on (X, μ), let $\mathcal{A}$ be a non-atomic σ-subalgebra of MALG, with associated map π, such that E is generated by $\mathcal{A}$, and let F be the factor of E determined by $\mathcal{A}$. Then we have that

$$t([F]) \leq t([E]^{\mathcal{A}}) \leq t([F]) + t([E_\pi]).$$

If then F, E_π are aperiodic, we have $t([F]), t([E_\pi]) = 2$ (see [LeM, p. 263]), so $t([E]^{\mathcal{A}}) \leq 4$. We do not know if 4 here can be lowered to 2.

18. The space of graphs

Consider again a measure preserving countable Borel equivalence relation E on (X, μ). Denote by $Gr(E)$ the set of all (simple, undirected) Borel graphs G on X such that $G \subseteq E$, where again we identify two such graphs if they agree a.e.

For any $G \in Gr(E)$ and $T \in [E]$, let again

$$A_{T,G} = \{x \colon (x, T(x)) \in G\}$$

and define the strong topology on $Gr(E)$ as the one generated by the maps

$$G \mapsto A_{T,G},$$

$$Gr(E) \to \mathrm{MALG},$$

for $T \in [E]$.

Note that we have the obvious analog of Proposition 4.11 (in relation to a generating sequence for E) and the following analog of Lemma 4.12.

Lemma 18.1. *Let Γ be a group, $a \colon \Gamma \times X \to X$ an action of Γ on a set X and put $a(g, x) = g \cdot x$. Let E_a be the induced equivalence relation on X and let $G \subseteq E_a$ be a graph. For $g \in \Gamma$, let*

$$A^a_{g,G} = A_{g,G} = \{x \colon (x, g \cdot x) \in G\}.$$

Then

1. $A_{1,G} = \emptyset$,
2. $A_{g,G} \subseteq g^{-1} \cdot A_{g^{-1},G}$,
3. $A_{h,G} \cap \mathrm{Fix}(h^{-1}g) \subseteq A_{g,G}$,

where

$$\mathrm{Fix}(p) = \{x \colon p \cdot x = x\}.$$

Conversely, if $(A_g)_{g\in\Gamma}$ is a family of sets satisfying 1.-2. above, then the relation

$$xGy \iff \exists g(g \cdot x = y \vee x \in A_g)$$

defines a graph contained in E_a and if 3. also holds we have that $A_g = A_{g,G}$.

Therefore the proofs in Chapter 4 show that this topology is Polish. We will simply call it the **topology** of $Gr(E)$. For this topology we have the following:

$$\begin{aligned} G_n \to G &\iff \forall i(A_{T_i,G_n} \stackrel{\mathrm{MALG}}{\longrightarrow} A_{T_i,G}) \\ &\iff \forall T \in [E](A_{T,G_n} \stackrel{\mathrm{MALG}}{\longrightarrow} A_{T,G}) \\ &\iff \forall \varphi \in [[E]](A_{\varphi,G_n} \stackrel{\mathrm{MALG}}{\longrightarrow} A_{\varphi,G}), \end{aligned}$$

where, as usual, $(T_i)_{i\in\mathbb{N}}$ generates E and for $\varphi \in [[E]]), A_{\varphi,G} = \{x \in \mathrm{dom}(\varphi) \colon (x, \varphi(x)) \in G\}$. Again as in Chapter 4, we can also view $Gr(E)$ as a closed subspace of MALG_E with the induced topology. Note also that if $G_0 \subseteq G_1 \subseteq \dots, G = \bigcup_n G_n$, then $G_n \to G$ and similarly if $G_0 \supseteq G_1 \dots, G = \bigcap_n G_n$.

One can also define the weak topology on $Gr(E)$ as the topology generated by the maps $G \mapsto \mu(A_{T,G}), Gr(E) \to [0,1]$, for $T \in [E]$. Anush Tserunyan pointed out that the proof of Theorem 4.15 shows that this topology coincides with the above (strong) topology. Indeed, let $G_n \to G$ in the weak topology and let $T \in [E]$ be an involution. Let $A = A_{T,G}$, which is T-invariant, and let $S \in [E]$ agree with T on A and be equal to the identity in its complement. Then $A_{S,G_n} \subseteq A$ (since $x \in A_{S,G_n} \implies x \neq S(x)$) and so $\mu(A \setminus A_{S,G_n}) \to 0$, since $A_{S,G} = A$. Also $A_{T,G} \setminus A_{T,G_n} = A \setminus A_{S,G_n}$, so $\mu(A_{T,G} \setminus A_{T,G_n}) \to 0$. As in the proof of Theorem 4.15 this implies that $\mu(A_{T,G_n} \setminus A_{T,G}) \to 0$, so $A_{T,G_n} \stackrel{\mathrm{MALG}}{\longrightarrow} A_{T,G}$.

Remark 18.2. On the set of bounded degree graphs in $Gr(E)$ one can also define the metric

$$D(G,H) = M(G \bigtriangleup H) = \int |G(x) \bigtriangleup H(x)| d\mu(x)$$

(see Lovász [L, page 352]), where M is the measure on E defined in Section 4.4, (1), and $G(x) = \{y \colon (x,y) \in G\}$ is the set of neighbors of x in G.

This gives rise to another topology on this set of graphs, for which is easy to check that it is at least as strong as the relative topology inherited from $Gr(E)$ (i.e., contains the relative topology). However, even for graphs of degree at most 2, it is easy to see that it may be actually strictly stronger. For example, take E to be the equivalence relation generated by a free, measure preserving action of the free group $\mathbb{F}_\infty$, with infinitely many generators $a_0, a_1, \dots$. Let G_n be the graph induced by the action of a_n, let $\mathbb{F}_\infty = \{g_0, g_1, \dots\}$ and put $T_i(x) = g_i \cdot x$. Then $A_{T_i, G_n} = X$, if $g_i = a_n$, while $A_{T_i, G_n} = \emptyset$, if $g_i \neq a_n$. Thus G_n converges to the empty graph in $Gr(E)$ but it is discrete in the metric D.

However if we consider the set of all d-regular graphs, for fixed $d \geq 2$, the D-topology on *that* set agrees with its relative topology from $Gr(E)$. Indeed assume that G_n, G are d-regular and $G_n \to G$. Let $\varphi_1, \dots, \varphi_d \in ((E))$ be such that for each x, $\varphi_1(x), \dots, \varphi_d(x)$ are exactly the G-neighbors of x. Then for each $i \leq d$, $\mu(A_{\varphi_i, G_n}) \to \mu(A_{\varphi_i, G}) = 1$. So given $\epsilon > 0$, find N large enough so that for $n \geq N$, $\mu(A_{\varphi_i, G_n}) > 1 - \frac{\epsilon}{d}$. Since the $\varphi_1(x), \dots, \varphi_d(x)$ are distinct and G_n is d-regular, it follows that for $x \in \bigcap_{i \leq d} A_{\varphi_i, G_n}$,we have $(G_n)(x) = G(x)$ and thus $D(G_n, G) \leq 2d\epsilon$, i.e., $D(G_n, G) \to 0$.

We also have the following analog of Theorem 5.1.

Theorem 18.3. *Let $G_n, G \in Gr(E)$ and $G_n \to G$. Then for each i, there is an increasing sequence $n_0^{(i)} < n_1^{(i)} < \dots$, so that $(n_m^{(i+1)})_{m \in \mathbb{N}}$ is a subsequence of $(n_m^{(i)})_{m \in \mathbb{N}}$ and*

$$G = \bigcup_m \bigcap_{k \geq m} G_{n_k^{(m)}}.$$

Proof. Let $(\varphi_i)_{i \in \mathbb{N}}$ be a sequence in $[[E]]$ such that $(x, y) \in G \iff \exists i (\varphi_i(x) = y)$. Then repeat the proof of Theorem 5.1 to define $(n_m^{(i)})_{m \in \mathbb{N}}$ and show that $G \subseteq \bigcup_m \bigcap_{k \geq m} G_{n_k^{(m)}}$. For the converse again repeat the proof of Theorem 5.1 by showing that if $H = \bigcup_m \bigcap_{k \geq m} G_{n_k^{(m)}}$ and $(\psi_i)_{i \in \mathbb{N}}$ is a sequence in $[[E]]$ such that $(x, y) \in H \iff \exists i (\psi_i(x) = y)$, then for $x \in \mathrm{dom}(\psi_i)$, we have $(x, \psi_i(x)) \in G$. $\square$

Again as in the paragraph following the proof of Theorem 5.1, Le Maître in [LeM1], showed that one has the following stronger form:

Let $G_n, G \in Gr(E)$ and $G_n \to G$. Then there is an increasing sequence

$n_0 < n_1 < \dots$, *so that*

$$G = \bigcup_m \bigcap_{k \geq m} G_{n_k}.$$

For $\mathcal{G} \subseteq Gr(E)$ we define $\mathcal{G}_\uparrow, \mathcal{G}_\downarrow$ as in the case of equivalence relations. Then we have:

Theorem 18.4. *If $\mathcal{G} \subseteq Gr(E)$ is closed under finite intersections, then $\overline{\mathcal{G}} = (\mathcal{G}_\downarrow)_\uparrow$. In particular, if $\mathcal{G}$ is hereditary, $\overline{\mathcal{G}} = \mathcal{G}_\uparrow$.*

A locally countable Borel graph G on X is (μ-)**measure preserving** if any partial Borel isomorphism $\varphi\colon A \to B$ such that graph$(\varphi) \subseteq G$ is measure preserving. This is equivalent to saying that the equivalence relation generated by G (i.e., the equivalence relation whose equivalence classes are the connected components of G) is measure preserving. Denote by $\mathcal{GR}$ the set of all Borel locally countable, measure preserving graphs on (X, μ), where as usual we identify two such graphs if they agree a.e. Then $Gr(E) = \{G \in \mathcal{GR}\colon G \subseteq E\}$ and $\mathcal{GR} = \bigcup_{E \in \mathcal{E}} Gr(E)$. As in Theorem 6.1, we can see that if $E \subseteq F$, then $Gr(E)$ is a closed subset of $Gr(F)$ and the topology of $Gr(E)$ is the relative topology it inherits form $Gr(F)$. Thus, as in Chapter 6, we can define the topology on $\mathcal{GR}$ which is the topological union of the topologies on $Gr(E)$, $E \in \mathcal{E}$.

Remark 18.5. As a final comment, we mention that ultraproducts of graphs can be defined as in Chapter 16 using Lemma 18.1. Also the uniform topology on $Gr(E)$ can be defined as in Section 4.6.

19. More complexity calculations

For each $G \in Gr(E)$, let $G^* \in S(E)$ be the equivalence relation generated by G. The argument in the paragraph preceding Proposition 4.29 shows that the operation $G \in Gr(E) \mapsto G^* \in S(E)$ is not continuous. Indeed, in the notation used there, we can take $G_n = F_n \setminus id$ and $G = F \setminus id$. Then $H_n = G_n \cup G$ is decreasing and $\bigcap_n H_n = G$, so $H_n \to G$, while $H_n^* = E_S$ and $G^* = E_{S^3}$. By a proof similar to that of Proposition 4.29 we also have the following:

Proposition 19.1. *The map $Gr(E) \ni G \mapsto G^* \in S(E)$ is of Baire class 1.*

We call $G \in Gr(E)$ a **graphing** of E is $G^* = E$.

Theorem 19.2. *The set $\{G \in Gr(E) \colon G$ is a graphing of $E\}$ is G_δ in $Gr(E)$. If E is aperiodic, it is also dense in $Gr(E)$.*

Proof. That it is G_δ follows from the preceding proposition, since G is a graphing of E iff $G^* = E$.

Assume now E is aperiodic in order to show that $\{G \colon G$ is a graphing of $E\}$ is dense in $Gr(E)$. A typical basic open set in $Gr(E)$ has the form

$$U_{G_0, T_1, \dots, T_n, \epsilon} = \{G \in Gr(E) \colon \forall 1 \le i \le n \; (\mu(A_{T_i, G} \Delta A_{T_i, G_0}) < \epsilon)\},$$

where $G_0 \in Gr(E), T_1, \dots, T_n \in [E]$ and $\epsilon > 0$. We will show that any such set contains a graphing of E. Since E is aperiodic, let $S_1, S_2, \dots \in [E]$ be aperiodic with $E_{S_1, S_2, \dots} = E$ (see [K, 8.5]). For each $1 \le i \le n, 1 \le j < \infty, k \in \mathbb{Z}$, let $A_{i,j,k} = \{x \colon T_i(x) = S_j^k(x)\}$. For fixed i, j, the sets $\{A_{i,j,k}\}_{k=1}^\infty$ are pairwise disjoint, so for any $\delta > 0$ there is $N_0(i, j, \delta)$ such that $\mu(\bigcup_{|k| \ge N_0(i,j,\delta)} A_{i,j,k}) < \delta$. Let $M_0(j, \delta) = \max_{1 \le i \le n} N_0(i, j, \delta)$.

Now define for each $1 \le j < \infty$ a graph $G_j \in Gr(E)$ as follows:

$$(x, y) \in G_j \iff y = S_j^{\pm k}(x) \vee y = S_j^{\pm (k+1)}(x),$$

where $k = M_0(j, \delta_j)$ with $\delta_j = \frac{\epsilon}{2^{j+1}}$. Let $G = G_0 \cup \bigcup_{j=1}^{\infty} G_j$. We claim that $G \in U_{G_0, T_1, \dots, T_n, \epsilon}$ and G is a graphing of E.

(1) $G \in U_{G_0, T_1, \dots, T_n, \epsilon}$: Let $1 \le i \le n$. Then $A_{T_i, G} = A_{T_i, G_0} \cup \bigcup_{j=1}^{\infty} A_{T_i, G_j}$. If $x \in A_{T_i, G_j}$, then there is $|k| \ge M_0(j, \delta_j)$, with $T_i(x) = S_j^k(x)$, so $x \in \bigcup_{|k| \ge N_0(i,j,\delta_j)} A_{i,j,k}$, therefore $\mu(A_{T_i, G_j}) < \delta_j$ and so

$$\begin{aligned}\mu(A_{T_i,G} \Delta A_{T_1,G_0}) &\le \mu(\bigcup_{j=1}^{\infty} A_{T_i,G_j}) \\ &\le \sum_{j=1}^{\infty} \delta_j = \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

(2) $G^* = E$: It is enough to show that for $x \in X$ and $1 \le j < \infty$, we have $(x, S_j(x)) \in G_j^*$. Let $k = M_0(j, \delta_j)$. Then $(x, S_j^{k+1}(x)) \in G_j$ and $(S_j(x), S_j^{k+1}(x)) \in G_j$, so $(x, S_j(x)) \in G_j^*$. $\square$

As in Chapter 8, if $\mathcal{G} \subseteq \mathcal{GR}$ is a class of measure preserving locally countable Borel graphs and $E \in \mathcal{E}$, we let

$$\mathcal{G}_E = \mathcal{G} \cap Gr(E).$$

In particular, $\mathcal{GR}_E = Gr(E)$.

We call $G \in \mathcal{GR}$ **acyclic** if for (almost) all x, there is no sequence $x = x_0, x_1, x_2, \dots, x_n$, with $n \ge 2$, of distinct points with $(x_0, x_1) \in G, (x_1, x_2) \in G, \dots, (x_{n-1}, x_n) \in G, (x_n, x_0) \in G$. We denote by $\mathcal{TR}$ the class of acyclic graphs.

Theorem 19.3. *The set $\mathcal{TR}_E = \{G \in Gr(E) \colon G$ is acyclic$\}$ is closed in $Gr(E)$.*

Proof. This follows from Theorem 18.4, but here is also a direct proof. Note that G is not acyclic iff $\exists n \ge 2 \exists T_1, T_2, \dots, T_n \in [E](\mu(\{x \colon \text{For all } 0 \le i < j \le n(x \notin \text{Fix}(T_i^{-1}T_j)) \ \& \ \forall i < n(x \in A_{T_i, T_{i+1}, G}) \ \& \ x \in A_{T_n, T_0, G})\}) > 0)$ where $T_0 = id$ and for $T \in [E], \text{Fix}(T) = \{x \colon T(x) = x\}$, so

$$\begin{aligned}&\{G \colon G \text{ is not acyclic}\} = \\ &\bigcup_{n \ge 2} \bigcup_{T_1, \dots, T_n \in [E]} \{G \colon \mu(\bigcap_{0 \le i < j \le n} (X \setminus \text{Fix}(T_i^{-1}T_j)) \cap \\ &\bigcap_{i<n} A_{T_i, T_{i+1}, G} \cap A_{T_n, T_0, G}) > 0\},\end{aligned}$$

which is clearly open. $\square$

A **treeing** G of E is an acyclic graphing of E.

Corollary 19.4. *The set $\{G \in Gr(E) \colon G$ is a treeing of $E\}$ is a G_δ set in $Gr(E)$.*

Similarly we define what it means to say that $G \in Gr(E)$ is a graphing of $F \in S(E)$ (namely $G^* = F$) or a treeing of F. We thus have:

Corollary 19.5. *The set $\{(G, F) \colon G$ is a graphing of $F\}$ is G_δ in $Gr(E) \times S(E)$. Similarly for $\{(G, F) \colon G$ is a treeing of $F\}$. In particular $\{F \in S(E) \colon F$ is treeable$\}$ is analytic in $S(E)$.*

Proof. Let $\{U_n\}$ be a countable open basis for $S(E)$. Then $G^* = F \iff \forall n(G^* \in U_n \Rightarrow F \in U_n)$. □

The following is a basic open problem.

Problem 19.6. *Is $\{F \in S(E) \colon F$ is treeable$\}$ Borel? Is there a Borel function $f \colon \{F \in S(E) \colon F \text{ is treeable}\} \to Gr(E)$ such that $f(F)$ is a treeing of F, if F is treeable.*

We next have the following fact, where for each $d \geq 1$, we let $\mathcal{GR}_d = \{G \in \mathcal{GR} \colon G \text{ has degree} \leq d\}$.

Proposition 19.7. *The set $\mathcal{GR}_{d,E} = \{G \in Gr(E) \colon G \text{ has degree} \leq d\}$ is closed in $Gr(E)$, for any $d \geq 1$.*

Proof. Again this follows from Theorem 18.4, but we can also give a direct proof. Note that

$$\begin{aligned} Gr(E) \setminus \{G \colon \text{ has degree} \leq d\} = \bigcup_{T_1,\dots,T_{d+1} \in [E]} &\{G \colon \\ &\mu(\bigcap_{1 \leq i < j \leq d+1} (X \setminus \mathrm{Fix}(T_i^{-1}T_j)) \\ &\cap \bigcap_{1 \leq i \leq d+1} A_{T_i, G}) > 0\}. \end{aligned}$$

□

Now let $\mathcal{BDG} = \{G \in \mathcal{GR} \colon G \text{ has bounded degree}\}$.

Corollary 19.8. *The set $\mathcal{BDG}_E = \{G \in Gr(E) \colon G$ has bounded degree$\}$ is dense F_σ in $Gr(E)$. Moreover, if E is aperiodic, then its complement is dense in $Gr(E)$, so $\mathcal{BDG}_E$ is in $F_\sigma \setminus G_\delta$.*

Proof. It is clear that $\mathcal{BDG}_E$ is F_σ by Proposition 19.7. Density is also easy, since if (T_n) is a uniquely generating sequence of Borel involutions for E, then for any $G \in Gr(E)$, if $G_n = G \cap \bigcup_{k\leq n} \text{graph}(T_k)$, then $G_n \to G$. Finally if E is aperiodic, put $H_n = G \cup \bigcup_{k\geq n} \{(x,y)\colon x \neq y \ \&\ T_k(x) = y\}$. Then $H_n \to G$. □

We also have, letting $\mathcal{IDG} = \{G \in \mathcal{GR}\colon G$ has infinite degree$\}$:

Proposition 19.9. *The set $\mathcal{IDG}_E = \{G \in Gr(E)\colon G$ has infinite degree$\}$ is G_δ in $Gr(E)$ and, if E is aperiodic, it is dense in $Gr(E)$.*

Proof. Let $(T_i)_{i\in\mathbb{N}}$ be a generating sequence for E. Let also for each n, $\mathcal{D}_n = \{T_0, \ldots, T_{n-1}\}$. Finally let $deg_G(x)$ be the degree of x in G. We have

$$\{x\colon deg_G(x) \geq d\} = \bigcup_n \bigcup_{S_1,\ldots S_d \in \mathcal{D}_n} \left[\bigcap_{1\leq i<j\leq d} (X \setminus \text{Fix}(S_i^{-1}S_j)) \cap \bigcap_{1\leq i\leq d} A_{S_i,G} \right],$$

and

$$G \in \mathcal{IDG}_E \iff \forall d \geq 1 \forall \epsilon \in \mathbb{Q}^+ (\mu(\{x\colon deg_G(x) \geq d\}) > 1 - \epsilon)$$

therefore $G \in \mathcal{IDG}_E$ iff the following holds: $\forall d \geq 1 \forall \epsilon \in \mathbb{Q}^+ \exists n$

$$\mu\Bigg(\bigcup_{S_1,\ldots S_d \in \mathcal{D}_n} \left[\bigcap_{1\leq i<j\leq d} (X \setminus \text{Fix}(S_i^{-1}S_j)) \cap \bigcap_{1\leq i\leq d} A_{S_i,G} \right] \Bigg) > 1 - \epsilon$$

so $\mathcal{IDG}_E$ is in G_δ.

Density in case of aperiodic E follows from the proof of Corollary 19.8, since the graphs H_n defined there are in $\mathcal{IDG}_E$. □

Finally we have, letting $\mathcal{LFG} = \{G \in \mathcal{GR}\colon G$ is locally finite$\}$:

Proposition 19.10. *The set $\mathcal{LFG}_E = \{G \in Gr(E)\colon G$ is locally finite$\}$ is $F_{\sigma\delta}$ in $Gr(E)$. Both $\mathcal{LFG}_E$ and its complement are dense in $Gr(E)$, if E is aperiodic. Moreover if E is ergodic, $\mathcal{LFG}_E$ is in $F_{\sigma\delta} \setminus G_{\delta\sigma}$.*

Proof. Using the notation of the proof of Proposition 19.9 we have

$$\begin{aligned} G \text{ is locally finite } &\iff \forall \epsilon \in \mathbb{Q}^+ \exists d \in \mathbb{N} \\ &\qquad \mu(\{x\colon deg_G(x) < d\}) \geq 1-\epsilon, \end{aligned}$$

so it is enough to show that for each fixed $d \in \mathbb{N}$,

$$\{G\colon \mu(\{x\colon deg(x) < d\}) \geq 1-\epsilon\}$$

is closed. Note that

$$\begin{aligned} \{x\colon deg_G(x) < d\} = &\bigcap_n \bigcap_{S_1,\dots,S_d \in \mathcal{D}_n} \\ &\left[\bigcup_{1\leq i<j\leq d} \mathrm{Fix}(S_i^{-1}S_j) \cup \bigcup_{1\leq i\leq d} (X \setminus A_{S_i,G}) \right], \end{aligned}$$

and

$$\begin{aligned} &\mu(\{x\colon deg_G(x)) < d\}) \geq 1-\epsilon \\ &\qquad \iff \forall n \Bigg[\mu\Bigg(\bigcap_{S_n,\dots,S_d \in \mathcal{D}_n} \Bigg[\bigcup_{1\leq i<j\leq d} \mathrm{Fix}(S_i^{-1}S_j) \cup \\ &\qquad\qquad \bigcup_{1\leq i\leq d} (X \setminus A_{S_i,G} \Bigg]\Bigg) \geq 1-\epsilon \Bigg], \end{aligned}$$

which is clearly a closed condition on G.

By Corollary 19.8 and Proposition 19.9 both $\mathcal{LFG}_E$ and its complement are dense in $Gr(E)$, if E is aperiodic.

Assume now that E is ergodic. Then the argument in the proof of Theorem 8.6 shows that $\mathcal{LFG}_E$ is not in $G_{\delta\sigma}$. □

Denote by $C(G)$ the **cost** of G, i.e., $C(G) = \frac{1}{2}\int deg_G(x) d\mu(x) \in [0,\infty]$.

Proposition 19.11. *The function $G \in Gr(E) \mapsto C(G)$ is lower semicontinuous, i.e., for every $r \in \mathbb{R}$, $\{G \in Gr(E)\colon C(G) > r\}$ is open. In particular, $\{G \in Gr(E)\colon C(G) = \infty\}$ is G_δ.*

Proof. We will show that for each $r \in \mathbb{R}$, $\{G \in Gr(E)\colon C(G) \leq r\}$ is closed. This follows from Theorem 18.4, since $G \subseteq H \Rightarrow C(G) \leq C(H)$ and $G_0 \subseteq G_1 \subseteq \cdots \Rightarrow C(\bigcup_n G_n) = \lim_{n\to\infty} C(G_n)$. □

Theorem 19.12. *If E is aperiodic, the set $\{G \in Gr(E) \colon C(G) = \infty\}$ is dense and therefore the generic $G \in Gr(E)$ is a graphing of E of infinite cost.*

Proof. Recall that E admits a measure M defined by

$$M(A) = \int |A_x| d\mu(x) = \int |A^y| d\mu(y),$$

for Borel $A \subseteq E$. Moreover for $G \in Gr(E)$, $C(G) = \frac{1}{2}M(G)$. Since E is aperiodic $M(E) = \infty$ and since $E = \bigcup_n \text{graph}(f_n)$ for Borel functions $f_n \colon X \to X, M$ is σ-finite.

Let $G \in Gr(E)$ in order to show that there is a sequence $G_n \in Gr(E)$ with $C(G_n) = \infty$ and $G_n \to G$. We can assume of course that $C(G) < \infty$.

Write $E \setminus G = \bigsqcup_n A_n$, with $M(A_n) < \infty$. Let $B_n = G \cup \bigsqcup_{m \geq n} A_m$, so that $M(B_n) = \infty, B_0 \supseteq B_1 \supseteq \dots, \bigcap_n B_n = G$. Let $G_n = (B_n \cup B_n') \setminus \{(x,x) \colon x \in X\}$, where $B_n' = \{(x,y) \in E \colon (y,x) \in B_n\}$. Then $G_n \in Gr(E), M(G_n) = \infty, G_0 \supseteq G_1 \supseteq G_2, \dots$ and moreover $\bigcap_n G_n = G$. Because if $(x,y) \in \bigcap_n G_n$, then $x \neq y$, and for infinitely many n, $(x,y) \in B_n$ or for infinitely many n, $(x,y) \in B_n'$, i.e., $(y,x) \in B_n$, so $(x,y) \in G$. Then $G_n \to G$ and we are done. □

It is also clear that the F_σ set $\{G \in Gr(E) \colon C(G) < \infty\}$ is dense, since every G can be written as the union of an increasing sequence G_n with $C(G_n) = \frac{1}{2}M(G_n) < \infty$. In particular, it follows that $\{G \in Gr(E) \colon C(G) < \infty\}$ is in $F_\sigma \setminus G_\delta$.

Finally we have the following result concerning locally finite graphings of equivalence relations (see [JKL, Theorem 3.12]).

Proposition 19.13. *There is a Borel function $\Lambda \colon S(E) \to Gr(E)$ such that for any $F \in S(E), \Lambda(F)$ is a locally finite graphing of F.*

Proof. Use Proposition 4.18 and [JKL, proof of Theorem 3.12]. □

20. Treeability

In view of the proof of Theorem 14.1 (in the ergodic case), one can try to approach Problem 19.6 by first trying to show an analog of Sublemma 14.3 for treeability. Recall that, by Theorem 19.3, the set

$$\mathcal{TR}_E = \{G \in Gr(E) \colon G \text{ is acyclic}\}.$$

is closed in $Gr(E)$. By Corollary 19.4, the set

$$Treeing(E) = \{G \in \mathcal{TR}_E \colon G \text{ is a treeing of } E\}$$

is G_δ in $\mathcal{TR}_E$. If E is not treeable, clearly this set is empty.

Problem 20.1. *If E is ergodic, treeable, is $Treeing(E)$ dense in $\mathcal{TR}_E$?*

We first note the following:

Proposition 20.2. *If there is $G \in \mathcal{TR}_E$ with $C(G) = \infty$, then*

$$\{G \in \mathcal{TR}_E \colon C(G) = \infty\}$$

is dense in $\mathcal{TR}_E$.

Proof. The proof is analogous to that of Theorem 19.12. Let $G \in \mathcal{TR}_E$ with $C(G) < \infty$. Fix $G_\infty \in \mathcal{TR}_E$ with $C(G_\infty) = \infty$. Let $S = G_\infty \setminus G$, so that $C(G) = \infty$. Write $S = \bigsqcup_n G_n$, with $G_n \in \mathcal{TR}_E$ and $C(G_n) < \infty$. Let $H_n = G \sqcup \bigsqcup_{m \geq n} G_m$. Then $C(H_n) = \infty$ and $H_0 \supseteq H_1 \supseteq \dots, \bigcap_n H_n = G$, so $H_n \to G$. □

Recall that $C(E)$ denotes the cost of the equivalence relation E.

Proposition 20.3. *Let E be ergodic with $C(E) > 1$. Then the set*

$$\{G \in \mathcal{TR}_E \colon C(G) = \infty\}$$

is dense G_δ in $\mathcal{TR}_E$. In particular if $1 < C(E) < \infty$, then the generic $G \in \mathcal{TR}_E$ is **not** *a treeing of E.*

Proof. By the proof of [KM, 28.8], since $C(E) > 1$, there is a free action of $\mathbb{F}_2$ on X whose equivalence relation is contained in E. Since $\mathbb{F}_\infty \subseteq \mathbb{F}_2$, this gives a $G \in \mathcal{TR}_E$ with $C(G) = \infty$. $\square$

Thus Problem 20.1 has a negative answer if $C(E) < \infty$, but E is not hyperfinite (in which case $C(E) > 1$). We next show that it has a positive answer if E is hyperfinite.

Proposition 20.4. *Let E be ergodic, hyperfinite. Then $Treeing(E)$ is dense in $\mathcal{TR}_E$ and thus the generic $G \in \mathcal{TR}_E$ is a treeing of E.*

Proof. Let $G_0 \in \mathcal{TR}_E, T_1, \dots, T_n \in [E]$ and $\epsilon > 0$. We need to find $G \in \mathcal{TR}_E$ which is a treeing of E and $\forall 1 \leq i \leq n(\mu(A_{T_i,G} \Delta A_{T_i,G_0}) < \epsilon)$.

First we claim that we can assume that G_0^* is a finite equivalence relation. Indeed $G_0^* \subseteq E$ is hyperfinite, so we can write $G_0^* = \bigcup_{n=1}^\infty E_n$, where $E_1 \subseteq E_2 \subseteq \dots$ and each E_n is finite. Let $G_i = E_i \cap G_0$, for $i = 1, 2, \dots$. Then $G_i \in \mathcal{TR}_E, G_i^* \subseteq E_i$ is finite, $G_1 \subseteq G_2 \subseteq \dots$ and $G_0 = \bigcup_{n=1}^\infty G_n$, so $G_n \to G_0$.

Since E is aperiodic, it is clear that every E-class contains infinitely many G_0^*-classes. Let $Y \subseteq X$ be a Borel transversal for G_0^*. Clearly $\mu(Y) > 0$ and Y meets every E-class infinitely often.

We claim that there is $T \in [E|Y]$ such that T generates $E|Y$ and moreover $\mu(\{x \in Y \colon \exists 1 \leq i \leq n(T(x) = T_i^{\pm 1}(x))\}) < \epsilon$. Granting this, let $G \in Gr(E)$ be defined by

$$(x, y) \in G \iff (x, y) \in G_0 \vee [x, y \in Y \ \&\ y = T^{\pm 1}(x)].$$

Then clearly $G \in \mathcal{TR}_E$ and $G^* = E$. Moreover for any $1 \leq i \leq n$,

$$A_{T_i,G} = A_{T_i,G_0} \sqcup \{x \in Y \colon T_i(x) = T^{\pm 1}(x)\},$$

thus

$$\mu(A_{T_i,G} \Delta A_{T_i,G_0}) = \mu(\{x \in Y \colon T_i(x) = T^{\pm 1}(x)\} < \epsilon.$$

It remains to prove the claim. It is enough to show that there is an aperiodic $S \in [E|Y]$ such that for each $1 \leq i \leq n, \mu(\{x \in Y \colon S(x) = T_i^{\pm 1}(x))\}) < \frac{\epsilon}{2n}$, so that $\mu(\{x \in Y \colon \exists 1 \leq i \leq n(S(x) = T_i^{\pm 1}(x))\}) < \frac{\epsilon}{2}$. Because then applying the Conjugacy Lemma (see, [K, 3.4]) to $[E|Y]$, we can find $T \in [E|Y]$ which generates $E|Y$ and $\mu(\{x \in Y \colon S(x) \neq T(x)\}) < \frac{\epsilon}{2}$ and thus

$$\mu(\{x \in Y \colon \exists 1 \leq i \leq n(T(x) = T_i^{\pm 1}(x))\}) < \epsilon.$$

To find S, let $S_0 \in [E|Y]$ be aperiodic and generate $E|Y$. Let for each $1 \leq i \leq n$,

$$Z_i = \{x \in Y : T_i^{\pm 1}(x) \in Y\}.$$

If $x \in Z_i$, then $(x, T_i^{\pm 1}(x)) \in E|Y$, so there is some $m \in \mathbb{Z}$ with $T_i^{\pm 1}(x) = S_0^m(x)$. Let for $N \in \mathbb{N}$,

$$Z_{N,i} = \{x \in Y : \exists |m| \leq N(T_i^{\pm 1}(x) = S_0^m(x))\}.$$

Then $Z_{0,i} \subseteq Z_{1,i} \subseteq \dots$ and $\bigcup_N Z_{N,i} = Z_i$. So find N_0 large enough with

$$\mu(Z_i \setminus Z_{N_0,i}) < \frac{\epsilon}{2n}.$$

Let $S = S_0^{N_0+1} \in [E|Y]$, which is clearly aperiodic. If $x \in Y$ and $S(x) = T_i^{\pm 1}(x)$, then $T_i^{\pm 1}(x) = S_0^{N_0+1}(x)$, so $x \in Z_i \setminus Z_{N_0,i}$, thus $\mu(\{x \in Y : S(x) = T_i^{\pm 1}(x)\}) \leq \mu(Z \setminus Z_{N_0}) < \frac{\epsilon}{2n}$, and the proof is complete. $\square$

Corollary 20.5. *Let E be ergodic, with finite cost. Then E is hyperfinite iff the generic $G \in \mathcal{TR}_E$ is a treeing of E.*

Thus the only remaining open case of Problem 20.1 is when $C(E) = \infty$.

There is actually a strengthening of Proposition 20.4, proved by Anush Tserunyan, with a simpler proof than the above. We will use below the following notation and terminology,.

We call $G \in Gr(E)$ **finite, smooth, hyperfinite** if G^* is, resp., finite, smooth, hyperfinite. Let $\mathcal{FTR}_E$, $\mathcal{STR}_E$ and $\mathcal{HTR}_E$ denote, resp., the set of finite, smooth, hyperfinite $G \in \mathcal{TR}_E$. Note that by Theorem 18.4, $\mathcal{HTR}_E$ is closed and $\overline{\mathcal{FTR}_E} = \overline{\mathcal{STR}_E} = \mathcal{HTR}_E$.

Proposition 20.6 (Tserunyan)**.** *Let E be aperiodic and treeable. For any $G_0 \in \mathcal{STR}_E$ and $T_1, \dots, T_m \in [E]$, there is a treeing $G \supseteq G_0$ of E such that $A_{T_i,G} = A_{T_i,G_0}$, for all i.*

Proof. Let Y be a transversal for G_0^* and take a group $\Gamma = \{g_n\}_{n \in \mathbb{N}} \subseteq [E]$ that generates E. We first handle two special cases and the general case will follow from them.

Case 1: $[E : G_0^*] < \infty$. Then $E|Y$ is a finite equivalence relation and hence there is a Borel selector $s : Y \to Y$ for $E|Y$ such that $[s(y)]_{G_0^*}$ is infinite for all $y \in Y$ (such s exists because E is aperiodic). Now define a function $g : Y \setminus s(Y) \to X$ by $g(y) = g_n(y)$, where n is the least such that $g_n(y) \in$

$[s(y)]_{G_0^*}$ and for any $i = 1, ..., m$, $g_n(y) \neq T_i(y)$ and $T_i(g_n(y)) \neq y$ (such n exists since $[s(y)]_{G_0^*}$ is infinite). Finally, put $G = \{(y, g(y)), (g(y), y) : y \in Y \setminus s(Y)\} \cup G_0$. It is straightforward to check that G is a treeing of E satisfying the condition of the proposition.

Case 2: $[E : G_0^*] = \infty$. Let H be the graph on Y generated by $T_1, ..., T_m$, i.e. for $x, y \in Y$,

$$xHy \iff x \neq y \text{ and } \exists i(T_i(x) = y \text{ or } T_i(y) = x).$$

Since H is locally finite, it admits a Borel countable coloring (actually a finite coloring), and thus there is a maximal independent Borel subset Z. For every E-class C, $Y \cap C$ is infinite (by the condition of the case), and hence $Z \cap C$ is infinite as well because otherwise there would be a point in $(Y \setminus Z) \cap C$ independent from $Z \cap C$ in H, contradicting the maximality of Z. Thus we can define a function $g : Y \setminus Z \to Z$ by $g(y) = g_n(y)$, where n is the least such that $g_n(y) \in Z$ and for any $i = 1, ..., m$, $g_n(y) \neq T_i(y)$ and $T_i(g_n(y)) \neq y$. Put $G_1 = \{(y, g(y)), (g(y), y) : y \in Y \setminus Z\}$. Also, let G_2 be a treeing of $E|Z$ (which exists because E is treeable). Finally, put $G = G_0 \cup G_1 \cup G_2$. Again, it is not hard to check that G is a treeing of E satisfying the condition of the proposition.

General case: Let

$$X_1 = \{x \in X : [x]_E \text{ contains only finitely many } G_0^* \text{ classes}\}$$

and put

$$X_2 = X \setminus X_1.$$

Then combine the treeings for $E|X_1$ and $E|X_2$ provided by cases 1 and 2. □

Theorem 20.7 (Tserunyan)**.** *Let E be aperiodic and treeable. Then we have $\mathcal{HTR}_E \subseteq \overline{Treeing(E)}$. In particular, if E is hyperfinite, then $Treeing(E)$ is dense in $\mathcal{TR}_E$.*

Proof. Fix $G_0 \in \mathcal{HTR}_E$. Since G_0 is hyperfinite, we have $G_0^* = \bigcup_{n \geq 1} E_n$, where E_n are increasing and finite. Letting $G_n = E_n \cap G_0$, we get $G_0 = \bigcup_{n \geq 1} G_n$ and thus $G_n \to G_0$. By Proposition 20.6, G_n is in $\overline{Treeing(E)}$, and hence so is G_0. □

Remark 20.8. Note that Proposition 20.6 cannot be extended to graphs $G_0 \in \mathcal{HTR}_E$, even if we drop the requirement about the T_i's. Indeed, let E be aperiodic, hyperfinite and F a proper aperiodic, hyperfinite subequivalence relation of E. If G_0 is a treeing of F, then it cannot be extended to a treeing G of E, since then the cost of G would be bigger than the cost of G_0, contradicting the fact that they are both equal to 1.

Let $SubTreeing(E)$ denote the set of all graphs in $\mathcal{TR}_E$ that are contained in treeings of E, i.e.,

$$SubTreeing(E) = \{G_0 \in \mathcal{TR}_E : \exists G \in Treeing(E)(G \supseteq G_0)\}.$$

Proposition 20.9 (Tserunyan)**.** *Let E be treeable. Then*

$$\mathcal{STR}_E \subseteq SubTreeing(E).$$

Therefore, in particular, we have $\mathcal{HTR}_E \subseteq \overline{SubTreeing(E)}$.

Proof. Fix $G_0 \in \mathcal{STR}_E$ and let $Y \subseteq X$ be a Borel transversal for G_0^*. Since E is treeable, there is a treeing G of $E|Y$. It is clear that $G_0 \cup G$ is a treeing of E. □

Proposition 20.10. *Let $G_0 \in SubTreeing(E)$.*

(a) (Tserunyan) *For any Borel set $A \subseteq X$, $G_0|A \in SubTreeing(E|A)$.*

(b) (Conley) *For a Borel equivalence relation F with $G_0^* \subseteq F \subseteq E$, $G_0 \in SubTreeing(F)$.*

Proof. Let $G \supseteq G_0$ be a treeing of E. For (a), project G onto $G|A$ as described in the proof of [JKL, Proposition 3.3 (i)] to get $G' \in Treeing(E|A)$ such that $G' \supseteq G|A$. Similarly, for (b), use the same method to project G onto $G|C$, for each F-class C, to get $G' \in Treeing(F)$ with $G' \supseteq (G \cap F) \supseteq G_0$. One could also note that (a) follows from (b). □

Theorem 20.11 (Tserunyan)**.** *For any $G_0 \in SubTreeing(E)$ and automorphisms $T_1, ..., T_m \in [E]$, there is a treeing $G \supseteq G_0$ of E such that $A_{T_i,G} = A_{T_i,G_0}$, for all i. In particular, $Treeing(E)$ is dense in $SubTreeing(E)$ and hence $\overline{Treeing(E)} = \overline{SubTreeing(E)}$.*

Proof. Let X_0 be the E-saturation of $\{x \in X : [x]_{G_0^*} \text{ is infinite}\}$. Then X_0 is E-invariant and $G_0^*|(X \setminus X_0)$ is a finite equivalence relation, so Proposition 20.6 applies to $G_0|(X \setminus X_0)$, and we may assume that $X_0 = X$. Thus each E-class C contains an infinite G_0^*-class and hence

$$A := \{x \in X : [x]_{G_0^*} \text{ is infinite}\}$$

is a complete section. Note that $B := X \setminus A$ is G_0^*-invariant and $G_0^*|B$ is a finite equivalence relation. Let Y be a transversal for $G_0^*|B$ and fix a group $\Gamma = \{g_n\}_{n\in\mathbb{N}} \subseteq [E]$ that generates E. Define $g : Y \to A$ by $g(y) = g_n(y)$, where n is the least such that $g_n(y) \in A$ and for any $i = 1, ..., m$, $g_n(y) \neq T_i(y)$ and $T_i(g_n(y)) \neq y$ (such n exists since $[y]_E \cap A$ is infinite). Put $G_1 = \{(y, g(y)), (g(y), y) : y \in Y\}$.

We now construct a treeing $G_2 \supseteq G_0|A$ of $E|A$ such that $A_{T_i,G_2} = A_{T_i,G_0|A}$, for all i. By (a) of Proposition 20.10, there is $G' \in Treeing(E|A)$ such that $G' \supseteq G_0|A$. Fix a Borel linear ordering $<$ on X and define a function $f : G' \setminus G_0 \to E|A$ as follows: for $(x, y) \in G' \setminus G_0$, let $x' = \min_<(x, y)$ and $y' = \max_<(x, y)$, and put

$$f(x, y) = (x', g_n(x')),$$

where n is the least such that $g_n(x')G_0^*y'$ and for all $i = 1, ..., m$, $g_n(x') \neq T_i(x')$ and $T_i(g_n(x')) \neq x'$ (such n exists since $[y']_{G_0^*}$ is infinite). Now let G'' denote the symmetrization of $f(G' \setminus G_0)$ and put $G_2 = G_0|A \cup G''$. To see that G_2 is a treeing of $E|A$ note that $G_2 \supseteq G_0|A$ and for any two G_0-connected components $D_1, D_2 \subseteq A$, there is an edge between D_1 and D_2 in G_2 if and only if there is one in G' (in other words the projections of G_2 and G' on the quotient X/G_0^* coincide). Now it is clear that $G = G_0 \cup G_1 \cup G_2$ satisfies the condition of the lemma. $\square$

Let $MaxTr(E)$ denote the set of maximal (under inclusion) graphs in $\mathcal{TR}_E$; that is,

$$MaxTr(E) = \{G \in \mathcal{TR}_E : \forall G' \in \mathcal{TR}_E(G' \supseteq G \Rightarrow G' = G\}.$$

Theorem 20.12 (Tserunyan)**.** *Let E be a (not necessarily treeable) equivalence relation. Then for any $G_0 \in \mathcal{TR}_E$ and $T_1, ..., T_m \in [E]$, there is $G \in MaxTr(E)$ such that $G \supseteq G_0$ and $A_{T_i,G} = A_{T_i,G_0}$, for all i. In particular, $MaxTr(E)$ is dense in $\mathcal{TR}_E$.*

Proof. Let $G' \in MaxTr(E)$ with $G' \supseteq G_0$ (it exists since, modulo a null set, any increasing wellordered chain stabilizes in countably many steps). Put $F = (G')^*$ and note that $G_0 \in SubTreeing(F)$. Let $S_i \in [F], 1 \leq i \leq m$, be such that $(x, T_i(x)) \in F \implies S_i(x) = T_i(x)$, so that $A_{T_i,H} = A_{S_i,H}$, for any $H \in Gr(F)$. Applying Theorem 20.11 to F and $S_1, \ldots, S_m$ (in lieu of E and $T_1, \ldots, T_m$), we get $G \in Treeing(F)$ such that $G \supseteq G_0$ and $A_{T_i,G} =$

A_{T_i,G_0}, for all i. It remains to show that G is maximal. Let $G_1 \in \mathcal{TR}_E$ be such that $G_1 \supseteq G$. Note that because G and G' have the same connected components, $G' \cap (G_1 \setminus G) = \emptyset$ and $G_2 := G' \cup (G_1 \setminus G) \in \mathcal{TR}_E$. Thus, by the maximality of G', $G_2 = G'$. But $G_2 \setminus G' = G_1 \setminus G$ and hence, $G_1 = G$. $\square$

Let $SubTreeing^*(E)$ denote the set of all graphs in $SubTreeing(E)$ that are not treeings anywhere; i.e.

$$SubTreeing^*(E) = \{G \in SubTreeing(E) : \mu(\{x \in X : [x]_E = [x]_{G^*}\}) = 0\}.$$

Proposition 20.13 (Tserunyan). *For any $G \in Treeing(E)$ and $\epsilon > 0$, there is $G_0 \in SubTreeing^*(E)$ with $G_0 \subseteq G$ such that $\mu(A_{T,G} \setminus A_{T,G_0}) < \epsilon$, for all $T \in [E]$. In particular, $SubTreeing^*(E)$ is dense in $Treeing(E)$.*

Proof. Let $Y \subseteq X$ be a Borel complete section for E such that $\mu(Y) < \frac{\epsilon}{2}$ (which exists by the Marker Lemma, see [KM, 6.7]) and put

$$G_0 = \{(x, y) \in G : x, y \notin Y\}.$$

Clearly $G_0 \in SubTreeing^*(E)$, and for any $T \in [E]$,

$$A_{T,G} \setminus A_{T,G_0} \subseteq Y \cup T^{-1}(Y).$$

Thus $\mu(A_{T,G} \setminus A_{T,G_0}) \leq \mu(Y) + \mu(T^{-1}(Y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. $\square$

Theorem 20.11 and Proposition 20.13 together imply:

Theorem 20.14 (Tserunyan). $\overline{SubTreeing^*(E)} = \overline{Treeing(E)} = \overline{SubTreeing(E)}$.

Bibliography

[AE] M. Abért and G. Elek, The space of actions, partition metric and combinatorial rigidity, *arXiv:1108.21471v1.* [110]

[AW] M. Abért and B. Weiss, Bernoulli actions are weakly contained in any free action, *Ergodic Theory Dynam. Systems*, **33** (2013), 323–333. [99]

[AS] S.R. Adams and R.J. Spatzier, Kazhdan groups, cocycles and trees, *Amer. J. Math*, **112** (1990), 271–287 [64]

[BK] H. Becker and A.S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, Cambridge Univ. Press, 1996. [7]

[B] G. Beer, A Polish topology for the closed subsets of a Polish space, *Proc. Amer. Math. Soc.*, **113 (4)** (1991), 1123–1133. [7]

[Bo] L. Bowen, Every countably infinite group is almost Ornstein, *Contemp. Math.*, **567** (2012), 67–78. [25]

[Bo1] L. Bowen, Simple and large equivalence relations, *Proc. Amer. Math. Soc.*, **145(1)** (2017), 215–224. [63, 65]

[BJ] C.M. Boykin and S. Jackson, Borel boundedness and the lattice rounding property, *Contemp. Math.*, **425** (2007), 113–126. [35, 36]

[CK] R. Chen and A.S. Kechris, Structurable equivalence relations, *Fund. Math.*, **242** (2018), 109–185. [91]

[CI] I. Chifan and A. Ioana, Ergodic subequivalence relations induced by a Bernoulli action, *Geom. Funct. Anal.*, **20** (2010), 53–67. [64]

[CKT] C.T. Conley, A.S. Kechris and R.D. Tucker-Drob, Ultraproducts of measure preserving actions and graph combinatorics, *Ergodic Theory Dynam. Systems*, **33**, (2013), 334–374. [97, 98, 101, 110, 111, 117]

[CM] C.T. Conley and B.D. Miller, An antibasis theorem for graphs of infinite Borel chromatic number, *Proc. Amer. Math. Soc.*, **142** (2014), 2123–2133. [41, 43, 70]

[CM1] C.T. Conley and B.D. Miller, Measure reducibility of countable Borel equivalence relations, *Ann. of Math.*, **185(2)** (2017), 347–402. [22]

[DC] Y. de Cornulier, A note on quotients of word hyperbolic groups with property (T), *arXiv:math/0504193v3*. [63]

[DJK] R. Dougherty, S. Jackson and A.S. Kechris, The structure of hyperfinite Borel equivalence relations, *Trans, Amer. Math. Soc.*, **341(1)** (1994), 193–225. [89]

[D] J. Dugundji, *Topology*, Allyn and Bacon, 1966. [38]

[EL] G.Elek and G. Lippner, Sofic equivalence relations, *J. Funct. Anal.*, **258** (2010), 1692–1708. [117]

[F] R.H. Farrell, Representation of invariant measures, *Illinois. J. Math.*, **6** (1962), 447-467. [24]

[FSZ] J. Feldman, C.E. Sutherland and R.J. Zimmer, Subrelations of ergodic equivalence relations, *Ergodic Theory Dynam. Systems*, **9** (1989), 239–269. [63, 79, 80, 111]

[FW] M. Foreman and B. Weiss, An anti-classification theorem for ergodic measure preserving transformations, *J. Eur. Math. Soc.*, **6** (2004), 277–292. [50]

[G1] D. Gaboriau, Invariants L^2 de relations d'equivalence et des groupes, *Publ. Math. Inst. Hautes Études Sci.*, **95** (2002), 93–150. [59]

[G2] D. Gaboriau, Orbit equivalence and measured group theory, *Proc. Intern. Congress of Math.*, Vol. III, Hindustan Book Agency, 1501–1527, 2010. [60, 64]

[GL] D. Gaboriau and R. Lyons, A measurable-group-theoretic solution to von Neumann's problem, *Invent. Math.*, **177(3)** (2009), 533–540. [64]

[GT] D. Gaboriau and R.D. Tucker-Drob, Approximations of standard equivalence relations and Bernoulli percolation at p_u, *C.R. Acad. Sci. Paris Sér. I Math.*, **354(11)** (2016), 1114–1118. [69]

[Gl] E. Glasner, *Ergodic Theory via Joinings*, Amer. Math. Soc., 2003. [114]

[GW] E. Glasner and B. Weiss, Kazhdan's property T and the geometry of the collection of invariant measures, *Geom. Funct. Anal.*, **7** (1997), 917–935. [96]

[H] L. Harrington, Analytic determinacy and $0^\#$, *J. Symb. Logic*, **43(4)** (1978), 685–693. [44]

[HK] G. Hjorth and A.S. Kechris, Rigidity theorems for actions of product groups and countable Borel equivalence relations, *Memoirs Amer. Math. Soc.*, **177 (833)** (2005). [59, 60, 68, 72]

[IKT] A. Ioana, A.S. Kechris, and T. Tsankov, Subequivalence relations and positive-definite functions, *Groups Geom. Dyn.*, **3** (2009), 579–625. [14, 69]

[JKL] S. Jackson, A.S. Kechris, and A. Louveau, Countable Borel equivalence relations, *J. Math. Logic*, **2 (1)** (2002), 1–80. [134, 139]

[K] A.S. Kechris, *Global aspects of ergodic group actions*, Amer. Math. Soc., 2010. [5, 10, 15, 28, 38, 42, 45, 50, 51, 52, 56, 58, 68, 75, 76, 81, 85, 86, 88, 97, 99, 113, 114, 129, 136]

[K1] A.S. Kechris, Weak containment in the space of actions of a free group, *Israel J. Math.*, **189** (2012), 461–507. [28]

[K2] A.S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995. [8, 43, 58, 87, 88, 104]

[K3] A.S. Kechris, Global aspects of measure preserving equivalence relations and graphs, *New Zealand J. Math.*, **52** (2021), 691–726. [6]

[K4] A.S. Kechris, The theory of countable Borel equivalence relations, *preprint*, posted in pma.caltech.edu/people/alexander-kechris [17, 30, 32, 89]

[KM] A.S. Kechris and B.D. Miller, *Topics in Orbit Equivalence*, Springer, 2004. [29, 50, 56, 57, 73, 74, 75, 91, 106, 109, 110, 122, 123, 136, 141]

[KST] A.S. Kechris, S. Solecki and S. Todorcevic, Borel chromatic numbers, *Adv. Math.*, **141** (1999), 1–44. [17, 116]

[KT] J. Kittrell and T. Tsankov, Topological properties of full groups, *Ergodic Theory Dynam. Systems*, **30** (2010), 525–545. [10, 103]

[LeM] F. Le Maître, The number of topological generators for full groups of ergodic equivalence relations, *Invent. Math.*, **198(2)** (2014), 261–268. [124]

[LeM1] F. Le Maître, On dense orbits on the space of subequivalence relations, *arXiv:2405.01806v1*. [15, 16, 34, 59, 62, 127]

[L] L. Lovász, *Large Networks and Graph Limits*, Amer. Math. Soc., 2012. [126]

[M] B.D. Miller, Incomparable treeable equivalence relations, *J. Math. Logic*, **12(1)** (2012), 125004-1–125004-11. [68]

[M1] B.D. Miller, Full groups, classification, and equivalence relations, *Ph.D. Thesis*, U.C. Berkeley, 2004. [10]

[M2] B.D. Miller, Composition of periodic automorphisms, *preprint*, posted in glimmeffros.github.io under Recent. [10]

[O] Yu. A. Olshanskii, On residualing homomorphisms and G-subgroups of hyperbolic groups. *Internat. J. Algebra Comput.*, **3(4)** (1993), 365–409. [63]

[Po] S. Popa, Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups, *Invent. Math.*, **170(2)** (2007), 243–295. [63, 65]

[S] R.M. Solovay, The independence of DC from AD, *Cabal Seminar 76-77*, Ed. by A.S. Kechris and Y.N. Moschovakis, Lecture Notes in Mathematics, No. 689, Springer, 1978, 171–184. [44]

[T] A. D. Törnquist, The Borel complexity of orbit equivalence, *Ph.D. Thesis*, UCLA, 2005. [43]

[T-D] R.D. Tucker-Drob, Shift-minimal groups, fixed price 1, and the unique trace property, *arXiv:1211.6395v3.* [91, 93]

[Va] S. Vaes, Explicit computations of all finite index bimodules for a family of II_1 factors, *Ann. Scient. Ecole Normal Sup.*, **4(1)** (2008), 743–788. [65]

[V] V.S. Varadarajan, Groups of automorphisms of Borel spaces, *Trans. Amer. Math. Soc.*, **109** (1963), 191–220. [24]

Index