Sets that can tile Lattices

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1 Introduction

In this paper, we study the properties of partitions of the natural numbers into multiples of a finite set of integers. The main question of interest is the following:

Theorem 1. *Let $a_1 < a_2 < \cdots < a_n$ be positive integers such that the set of positive integers can be partitioned into an infinite number of sets, each of the form $\{a_1k, a_2k, \ldots, a_nk\}$ for some positive integer k. Then, $a_i \mid a_n$ for all $1 \leqslant i \leqslant n$.*

Such partitions of the naturals involve only the multiplicative structure of $\mathbb{N}$. By considering the prime factorization of integers, the theorem, in effect, can be recast into one about partitioning $\mathbb{N}_0^d$ (the set of d-dimensional lattice points with nonnegative coordinates) into disjoint translates of a finite subset. In particular, we will prove the following:

Theorem 2. *Suppose $\mathcal{S}$ is a finite subset of $\mathbb{N}_0^d$ such that there exists a partition $\mathbb{N}_0^d = \mathcal{S}_1 \sqcup \mathcal{S}_2 \sqcup \cdots$ such that for each $\mathcal{S}_i$, there exists $t_i \in \mathbb{N}_0^d$ such that $\mathcal{S}_i = \{s + t_i \mid s \in \mathcal{S}\}$. Then, there exists a unique maximal element s_n of $\mathcal{S}$, in the sense that every $s_i \in \mathcal{S}$ satisfies that the j^{th} coordinate of s_i is at most the j^{th} coordinate of s_n for each $1 \leqslant j \leqslant d$.*

To see that Theorem 1 follows from Theorem 2, we let $\{p_1, \ldots, p_d\}$ be the set of primes that divide at least one of $\{a_1, \ldots, a_n\}$. Then, the prime factorizations of $\{a_1, \ldots, a_n\}$ determine a set $\mathcal{S}$ of n elements of $\mathbb{N}_0^d$ where each element s_i is given by the exponents of $p_1, \ldots, p_d$ in the prime factorization of a_i. Now suppose that $\{a_1, \ldots, a_n\}$ satisfies that a collection of sets of the form $\{a_1k, \ldots, a_nk\}$ partitions $\mathbb{N}$. Then, it is necessary and sufficient for there to exist a partition of the set $W = \{p_1^{c_1} \cdots p_d^{c_d} \mid (c_1, \ldots, c_d) \in \mathbb{N}_0^d\}$ into sets of the form $\{a_1k, \ldots, a_nk\}$. Each such set can be coordinatized as before, yielding a subset of $\mathbb{N}_0^d$ of the shape $\mathcal{S} + t := \{s + t \mid s \in \mathcal{S}\}$ which are translates of $\mathcal{S}$. Hence, $\mathcal{S}$ admits a partition of $\mathbb{N}_0^d$ via disjoint translates of $\mathcal{S}$. By Theorem 2, $\mathcal{S}$ has a unique maximal element, corresponding to a_n satisfying $a_i \mid a_n$ for all $1 \leqslant i \leqslant n$.

We shall now present several proofs of Theorem 2. The first is combinatorial in nature; the second uses a polynomial method. We then present another polynomial method, distinct from the first polynomial method but rich in ideas, that derives Theorem 2 for the case of $d = 2$.

2 Combinatorial Method

We will need the following definitions:

Definition 2.0.1 A *lattice* M spanned by vectors $b_1, \ldots, b_d \in \mathbb{N}_0^p$ is the set of points given by $e_1b_1 + \cdots + e_db_d$, where $e_1, \ldots, e_d$ are nonnegative integers. We denote $M = \text{span}(b_1, \ldots, b_d)$. If $b_1, \ldots, b_d$ are linearly independent, we say $b_1, \ldots, b_d$ form a basis of M.

For points $p = e_1b_1 + \cdots + e_db_d \in M$ where $b_1, \ldots, b_d$ are a basis of M, we coordinatize p as $(e_1, \ldots, e_d)$.

If $\mathcal{S} \subset M$ is such that disjoint translates of $\mathcal{S}$ can partition M, then we shall say that $\mathcal{S}$ is tileable. For tileable sets $\mathcal{S}$, we can express M as $M = \mathcal{S} \oplus \mathcal{T}$ where $\mathcal{T} \subset M$ is called the tiling. Also, for $a \in \mathcal{S}$ and $b \in M$, we shall say that b is tiled by a if $b \in \{a\} + \mathcal{T}$.

Definition 2.0.2. The *product order* is the partial ordering in M given by the following: for $p = (e_1, \ldots, e_d)$ and $q = (f_1, \ldots, f_d)$, we let $p \succeq q$ if $e_i \geqslant f_i$ for all $1 \leqslant i \leqslant d$.

Definition 2.0.3. For any set $\mathcal{S} \subset M$, we say that $a \in \mathcal{S}$ is a primitive element of $\mathcal{S}$ if a is a minimal element of $\mathcal{S} \setminus \{(0, \ldots, 0)\}$ with respect to the product order.

Definition 2.0.4. For a point $a = (a_1, \ldots, a_d)$ in a lattice M with basis $b_1, \ldots, b_d$, we shall define the sublattice $\mathrm{sp}(a) = \mathrm{span}(\{b_i \mid a_i > 0\})$, in other words, the span of all bases vectors whose coordinate in a is nonzero, and $\mathrm{sp}(a)^\perp = \mathrm{span}(\{b_i \mid a_i = 0\})$.

Observe that $\mathrm{sp}(a) \oplus \mathrm{sp}(a)^\perp = M$. This allows us to define a projection map $M \to \mathrm{sp}(a)$. We shall denote the projection of $c \in M$ as $\mathrm{proj}_{\mathrm{sp}(a)}(c)$.

2.1 L-Lemma

The central idea is that the tiling of primitive elements of tileable sets give us constraints on the tiling.

Theorem 3 (L-Lemma)**.** *Let $\mathcal{S}$ and $\mathcal{T}$ be (possibly infinite) sets and M be a lattice such that $\mathcal{S} \oplus \mathcal{T} = M$. If a is a primitive element of $\mathcal{S}$, then every element c of $\mathcal{S}$ satisfies $proj_{sp(a)}(c) = ma$ where m is a nonnegative integer.*

Proof. Call such points whose projection onto $\mathrm{sp}(a)$ is a multiple of a "regular." Assume for the sake of contradiction that not all elements of $\mathcal{S}$ are regular. Consider a minimal irregular $x \in \mathcal{S}$: that is, for every $y \prec x$, if $y \in \mathcal{S}$, then y is regular. Let B_x denote the set of $y \in M$ such that $y \prec x$. Notice that B_x is tiled by regular points. We are thus interested in how sets of regular points tile space.

Define L_a to be the set of points in $\mathrm{sp}(a)$ that are *not* $\succeq a$, and $\mathbb{N}_0 a$ to be the set of nonnegative integer multiples of a.

Proposition 2.1.1. $L_a \oplus \mathbb{N}_0 a = \mathrm{sp}(a)$.

Proof. We wish to show that each $p \in \mathrm{sp}(a)$ can be written uniquely as the sum of a multiple of a and an element of L_a. Indeed, let m be the largest nonnegative integer such that $ma \preceq p$. Then, $a \npreceq p - ma$, and thus p is the sum of ma and $p - ma \in L_a$. This representation is indeed unique: if $p = m'a + (p - m'a)$, then $m' \leqslant m$ by the maximality of m, and and if $m' < m$, then $(m - m')a \preceq p - m'a$ so $p - m'a \notin L_a$. □

It follows that $L_a \oplus \mathbb{N}_0 a \oplus \mathrm{sp}(a)^\perp = M$. This tells us that the set L_a tiles M with a certain tiling $\Psi = \mathbb{N}_0 a + \mathrm{sp}(a)^\perp$. Note that Ψ is the set of all regular points.

Claim. For each $\psi \in \Psi$, all points in $(L_a + \psi) \cap B_x$ are tiled by the same point in $\mathcal{S}$. In other words, for all $s_1, s_2 \in \mathcal{S}$, $t_1, t_2 \in \mathcal{T}$, and $\ell \in L_a$ satisfying $s_1 + t_1 = \psi$ and $s_2 + t_2 = \psi + \ell \prec x$, we have $s_1 = s_2$.

Proof. We induct on ψ consistent with product order. For the base case $\psi = 0$, note that because a is primitive, only 0 tiles $L_a \cap B_x$. For the induction step, assume the validity of the statement for all points $\psi' \prec \psi$; we shall show its validity for ψ.

Since $s_1 \preceq \psi \prec x$, we know that s_1 is regular. Thus, $t_1 = \psi - s_1 \in \Psi$. If $s_1 > 0$, then by the induction hypothesis, $t_1 + \ell$ is tiled by the same point in $\mathcal{S}$ as t_1, namely 0, so $t_1 + \ell \in \mathcal{T}$. Consequently, $\psi + \ell = s_1 + (t_1 + \ell)$ is the tiling of $\psi + \ell$, so $s_2 = s_1$, as desired. If $s_1 = 0$ but $s_2 \neq 0$, then $t_2 = \psi - s_2 + \ell \in T$ which implies, by the induction hypothesis, that $\psi - s_2 \in T$ and thus $s_2 + (\psi - s_2) = s_1 + t_1 = \psi$ are two ways of representing ψ as a sum of elements of $\mathcal{S}$ and $\mathcal{T}$, which is a contradiction, as desired. □

Having established what the tiling of points $\prec x$ "looks like," we now aim for a contradiction with the fact $x \in \mathcal{S}$. The general principle is the following: points that x tiles cannot be tiled by other elements of $\mathcal{S}$. To that end, let us say that $x \in \psi + L_a$ for some $\psi \in \Psi$. All points in $\psi + L_a$ and $\prec x$ are tiled by some $s \in \mathcal{S}$; note that s is regular.

Case 1: If $s \neq 0$, then $\psi - s \in T$ and $x - s \in (\psi - s) + L_a$. By the claim, $x - s$ is tiled by the same element of $\mathcal{S}$ as $\psi - s$, namely 0 (since $\psi - s \in \mathcal{T}$). Therefore, $x - s \in \mathcal{T}$ so x is tiled by s, reaching a contradiction for this case.

Case 2: If $s = 0$, let y be the (unique) minimum point satisfying $y \succeq x$ and $y \succeq \psi + a$. Notice that $(x + a) \succ x$ and $(x + a) \succ \psi + a$, so by the minimality of y, we have $y \prec x + a$. This implies that $y - x \prec a$, so $y - x \in \mathcal{T}$. Also, $\psi \preceq y - a \prec x$, so by the claim, $y - a$ is tiled by 0, i.e., $y - a \in \mathcal{T}$. We have derived that y is tiled by both x and a, a contradiction.

This completes the proof of the L-lemma. ■

2.2 Reduction

The L-lemma tells us that if $\mathcal{S}$ is tileable and $a \in \mathcal{S}$ is primitive, then all points in $\mathcal{S}$ are in the lattice M' spanned by a and the bases of $\mathrm{sp}(a)^\perp$. Note that M' is a sublattice of M. Let us coordinitize M' so that the first digit corresponds to the basis a, so that $a = (1, 0, \ldots, 0)$.

Proposition 2.2.1. $\mathcal{S}$ tiles M'.

Proof. Consider the tiling of M by $\mathcal{S}$. For any translated tile $\mathcal{S} + t$, either one of the following holds:

- If $t \in M'$ then $\mathcal{S} + t \subseteq M'$.
- If $t \notin M'$ then $(\mathcal{S}+)t \cap M' = \emptyset$.

Thus, all tiles that intersect M' are completely contained in M', so selecting these tiles yields a tiling of M'. □

We can write $M' = \mathcal{S} \oplus \mathcal{T}'$. Let k be the minimum positive integer such that $ka \notin \mathcal{S}$. Note ka is primitive in $M' \setminus \mathcal{S}$, so 0 must tile ka, and thus $ka \in \mathcal{T}'$. Applying the L-lemma on $\mathcal{T}'$ tells us that $\mathcal{T}'$ is contained in the lattice $M'_k = \mathrm{span}(ka, \mathrm{sp}(a)^\perp)$, so by Proposition 2.2.1, $\mathcal{T}'$ tiles M'_k.

Now we can write $\mathcal{T}' \oplus \mathcal{S}_k = M'_k$. Notice that $M'_k \oplus \{0, \ldots, k-1\} = M'$, so

$$\mathcal{T}' \oplus \mathcal{S}_k = M'_k \implies \mathcal{T}' \oplus \mathcal{S}_k \oplus \{0, \ldots, k-1\} = M' = \mathcal{T}' \oplus \mathcal{S} \implies \mathcal{S}_k \oplus \{0, \ldots, k-1\} = \mathcal{S}$$

We have thus shown that every tileable set $\mathcal{S}$ is expressible as $\mathcal{S}_k + \{0, \ldots, k-1\}$. This allows us to reduce $\mathcal{S}$ to $\mathcal{S}_k$, which is also a tileable set. Moreover, $|\mathcal{S}_k| = |\mathcal{S}|/k < \mathcal{S}$, so if we repeatedly apply reduction, we will eventually reduce our set to $\{0\}$.

We have in fact shown the following statement:

Theorem 4. *Every (finite) tileable $\mathcal{S}$ is expressible as $\{0, a_1, \ldots, k_1 a_1\} + \cdots + \{0, a_m \ldots, k_m a_m\}$.*

Note that theorem 2 follows, since the unique largest point in $\mathcal{S}$ is $k_1 a_1 + \cdots + k_m a_m$.

3 First Polynomial Method

For each point $(a_1, a_2, \ldots, a_d)$, we can associate it to the monomial $x_1^{a_1} \cdots x_d^{a_d}$. Then, the polynomial P associated with $\mathcal{S}$ is the sum of the monomials associated with the points in S.

Say a polynomial or formal power series is *unitary* if all its coefficients are 0 or 1. Notice that $P(x_1, \ldots, x_d)$ is unitary. Moreover, the power series $Q(x_1, \ldots, x_d)$ associated with our tiling T is also unitary. The condition that $\mathcal{S}$ tiles $\mathbb{N}_0^d$ with tiling $\mathcal{T}$ can be written as

$$P(x_1, \ldots, x_d) \cdot Q(x_1, \ldots, x_d) = \prod \frac{1}{(1 - x_i)}.$$

Remark that the fact that $\mathbb{Z}[[x_1, \ldots, x_d]]$ is a UFD means that the tiling $\mathcal{T}$ for each tileable $\mathcal{S}$ is unique. Now, $Q(x_1, \ldots, x_d)$ converges for $x_1, \ldots, x_d \in \mathbb{D}$ where $\mathbb{D}$ is the open unit disk, as does the right hand side $\prod \frac{1}{(1-x_i)}$. This implies that $P(x_1, \ldots, x_d)$ is nonzero for $x_1, \ldots, x_d \in \mathbb{D}$.

Now the trick is to consider $R(x) = P\left(x, x^N, x^{N^2}, \ldots, x^{N^{d-1}}\right)$ for a sufficiently large value of $N >> \deg(P)$. Notice that $R(x)$ does not have any roots in $\mathbb{D}$. Yet, $R(0) = 1$ (this is equivalent to the the origin belonging in the tile $\mathcal{S}$), so the product of the roots of $R(x)$ has magnitude 1. Consequently, all roots of $R(x)$ have magnitude 1. It follows that if z is a root of $R(x)$, then so is $\bar{z} = \frac{1}{z}$, and hence, R is symmetric: $R(x) = x^{\deg(R)} R\left(\frac{1}{x}\right)$.

As N is large, each term in P corresponds to a distinct term in $R(x)$. It follows that $P(x_1, \ldots, x_d)$ is also symmetric: $P(x_1, \ldots, x_d) = x_1^{a_1} \cdots x_d^{a_d} P\left(\frac{1}{x_1}, \ldots, \frac{1}{x_d}\right)$ for some integers $a_1, \ldots, a_d$. This shows that the tile $\mathcal{S}$ is symmetric, and in particular, it contains a maximal element, as desired.

4 Addendum: Second Polynomial Method

Each translation of S will have associated polynomial $x^a y^b S(x, y)$ for some a and b, so a partitioning of $\mathbb{N}^2$ by disjoint translations of S can be expressed by a unitary formal power series T such that

$$T(x, y) \cdot S(x, y) = \frac{1}{(1-x)(1-y)}, \tag{$\star$}$$

where T is a unitary formal power series. We establish the following claims:

Claim 4.1. For some integers k and ℓ and unitary polynomials $T_0(x)$ and $T_1(y)$ with $\deg T_0 < k$ and $\deg T_1 < \ell$,

$$S(x, 0) \cdot T_0(x) = \frac{x^k - 1}{x - 1} \quad \text{and} \quad S(0, y) \cdot T_1(x) = \frac{x^\ell - 1}{x - 1}.$$

Proof. We establish that $S(x,0)\cdot T_0(x)=\frac{x^k-1}{x-1}$, and the other case follows symmetrically.

From $(x,y)=(\frac{1}{2},0)$ in $(\star)$ we see that $T(\frac{1}{2},0)$ is rational, and thus the exponents in $T(x,0)$ must be eventually periodic with some period k. In particular, since $S(\frac{1}{2},0)$ is an integer over a power of two, it follows that $T(\frac{1}{2},0)$ is a power of two divided by an integer, so $T(x,0)$ is periodic. Thus we can write

$$T(x,0)=\frac{T_0(x)}{1-x^k}$$

for some integer k and unitary polynomial T_0. It then follows that

$$T_0(x,0)\cdot S(x,0)=\frac{x^k-1}{x-1},$$

as desired. □

Claim 4.2. If x_0 and y_0 are complex with $|x_0|,|y_0|<1$, then $S(x_0,y_0)\neq 0$.

Proof. Since T has all coefficients 0 or 1, it converges absolutely when $|x_0|,|y_0|<1$. It follows that

$$S(x_0,y_0)=\frac{1}{T(x_0,y_0)(1-x_0)(1-y_0)}\neq 0.$$

□

Express

$$\begin{aligned} T(x,y)&=T_0(x)+T_1(x)\cdot y+T_2(x)\cdot y^2+\cdots\\ \text{and}\quad S(x,y)&=S_0(x)+S_1(x)\cdot y+S_2(x)\cdot y^2+\cdots+S_n(x)\cdot y^n.\end{aligned}$$

Claim 4.3. For each t, we have $S_0\mid S_t$.

Proof. Let ζ be a root of $S(x,0)$. Since $S(x,0)$ divides $\frac{x^k-1}{x-1}$ for some k, we know $|\zeta|=1$. Now I contend $S(\zeta,y)=0$ for all y. If not, then $S(\zeta,y)=0$ has a root at $y=0$.

Perturb ζ slightly to ζ' where $|\zeta'|<1$, and consider complex-valued functions $f(y)=S(\zeta,y)$ and $g(y)=S(\zeta',y)-S(\zeta,y)$. There exists a ζ' such that $|g(y)|<|f(y)|$ for all $y\in\partial\mathbb{D}$; hence by Rouché's theorem, $f(y)=S(\zeta,y)$ and $f(y)+g(y)=S(\zeta',y)$ have the same number of roots in K. But $f(y)$ has a root $y=0$, so $S(\zeta',y)$ has at least one root with $|y|<1$, contradicting Claim 2.

Hence each root ζ of $S_0(x)$ is a root of $S_t(x)$ for all $t\geq 1$. Since S_0 divides $\frac{x^k-1}{x-1}$ and thus has no double root, it follows that $S_0(x)$ divides all $S_t(x)$. □

Claim 4.4. For each t, if $S_t\not\equiv 0$, we have $S_t(x)=S_0(x)\cdot x^{\bullet}$ for some $\bullet$.

Proof. We proceed by strong induction on t, with base case $t=0$ obvious.

Assume the hypothesis for all integers less than t, and assume $S_t\not\equiv 0$. The unitary formal power series $S_t(x)\cdot T_0(x)$ is always a polynomial multiple of $S_0(x)\cdot T_0(x)=\frac{1}{1-x}$; in particular if $R(x)=\frac{S_t(x)}{S_0(x)}=r_0+r_1x+\cdots+r_jx^j$ then

$$\begin{aligned} S_t(x)\cdot T_0(x)=\frac{R(x)}{1-x}&=\left(r_0+r_1x+r_2x^2+\cdots\right)\left(1+x+x^2+\cdots\right)\\ &=r_0+(r_0+r_1)x+(r_0+r_1+r_2)x^2+\cdots+(r_0+\cdots+r_j)(x^j+x^{j+1}+\cdots).\end{aligned}$$

Since $S_t(x)\cdot T_0(x)$ represents what S_t tiles in row t, it is unitary. In particular, the nonzero coefficients of R must alternate between $+1$ and -1, and since $R(1)=\frac{S_t(1)}{S_0(1)}>0$, we can express R as

$$R(x)=x^{R_0}-x^{R_1}+x^{R_2}-\cdots+x^{R_{2n}},\quad \text{where } R_0<R_1<\cdots<R_{2n}.$$

It also follows that

$$\frac{R(x)}{1-x} = \sum_{\substack{R_{2i} \le k < R_{2i+1} \\ \text{or } k \ge R_{2n}}} x^k \implies \frac{1-R(x)}{1-x} = \sum_{\substack{R_{2i-1} \le k < R_{2i} \\ \text{or } k < R_0}} x^k.$$

Now $\frac{R(x)}{1-x} = S_t(x) \cdot T_0(x)$ is the portion of row t covered by translations S_t, so $\frac{1-R(x)}{1-x}$ is the portion covered by $S_0, S_1, \ldots, S_{t-1}$, and since $S_0, S_1, \ldots, S_{t-1}$ are translations of S_0, this portion can be tiled by S_0. In particular, it is the product of S_0 and a unitary polynomial.

The key is to consider the polynomial

$$P(x) = \frac{R(x) \cdot (1 - R(x))}{1-x},$$

which is the product of $R(x) \cdot S_0(x) = S_t(x)$ and a unitary polynomial; in particular, it is a product of two unitary polynomials, hence its coefficients are all nonnegative.

However, consider the $x^{R_{2n}+R_{2n-1}-1}$ term of $P(x)$. In the product $P(x) = R(x) \cdot \frac{1-R(x)}{1-x}$, this term can only be obtained from a $-x^{R_{2n-1}}$ term from $R(x)$ and a $+x^{R_{2n}-1}$ term from $\frac{1-R(x)}{1-x}$. Thus the $x^{R_{2n}+R_{2n-1}-1}$ term in $P(x)$ has a coefficient of -1, contradiction. □

Thus the rows of $S(x, y)$ are translations of $S(x, 0)$, and similarly the columns are translations of $S(0, y)$. Consider the reduction

$$\widetilde{S}(x,y) = \frac{S(x,y)}{S(x,0) \cdot S(0,y)},$$

with the property that each row and column of $\widetilde{S}$ has at most one term. Since S tiles $\mathbb{N}^2$ and $\widetilde{S}$ tiles S, $\widetilde{S}$ tiles $\mathbb{N}^2$; let $\widetilde{T}(x,y) \cdot \widetilde{S}(x,y) = \frac{1}{(1-x)(1-y)}$.

If, for contradiction, $\widetilde{S}(x, y)$ contains monomials $x^a y^b$ and $x^c y^d$ where $a > c$ and $b < d$, then consider the monomial $x^a y^d$ in $\widetilde{S}(x,y) \cdot \widetilde{T}(x,y) = \frac{1}{(1-x)(1-y)}$. For one, $x^a y^d$ is contained in $x^a y^b \cdot \widetilde{T}(0, y)$, but for another it is contained in $x^c y^d \cdot \widetilde{T}(x, 0)$. Hence its coefficient is at least 2, contardiction.

It readily follows that $\widetilde{S}(x, y)$ has a top-right point, so $S(x, y) = S(x, 0) \cdot S(0, y) \cdot \widetilde{S}(x, y)$ has one as well. This completes the proof. ■

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