COMPUTATION AND DETECTION OF EVENTUALLY NEGLIGIBLE CLASSES IN MOD 2 COHOMOLOGY

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1. Introduction

Let G be a finite group, M a G-module, and F a field. A continuous group homomorphism $i : \Gamma_L = \mathrm{Gal}(L_{sep}/L) \to G$ yields a homomorphism $i^* : H^d(G,M) \to H^d(L,M)$ for every $d \in \mathbb{N}$. An element $u \in H^d(G,M)$ is *negligible over* F if $u \in \ker(i^*)$ for every i and every field extention L/F. If M is chosen so that one can form a cohomology ring via the cup product, an element is called *eventually negligible* if some power of it is negligible. It is known that the (eventually) negligible elements over F form a subgroup $H^d(G,M)_{\mathrm{neg}} \subset H^d(G,M)$ and that the (eventually) negligible classes of the cohomology ring $H^*(G,M)$ form an ideal [4]. However, it remains an open problem to find a precise description of (eventually) negligible elements for any given cohomology ring. In this paper, we compute the generators of the eventually negligible ideal for the cohomology of several types of groups with mod 2 coefficients.

1.1. **Notation.** D_{2n} is the dihedral group of order $2n$.
$N(G,\mathbb{Z}/2\mathbb{Z})$ is the mod 2 ideal of eventually negligible elements of G over $\mathbb{Q}$.
$I(G,\mathbb{Z}/2\mathbb{Z})$ is the mod 2 negligible ideal of G over $\mathbb{Q}$.

In Simon King and David Green's cohomology computations [6], generators are expressed in the form of a_{ij}, such that i,j are positive integers. The i indicates the degree (corresponds to $H^i(G,\mathbb{Z}/p\mathbb{Z})$), while j is used for indexing.
Alternatively, for the finite abelian group computations, we adopt the notation similar to those in Gherman's work [3]. The generators are in the form of x_i, y_i, such that x is a degree 1 generator, and y is a degree 2 generator. The subscript i is for indexing.

2. Objectives and Methods

We aim to classify all eventually negligible elements of the mod 2 cohomology of finite abelian, dihedral, semidihedral, modular, and quaternion 2-groups. First, we present some helpful definitions and results:

Definition 2.1. *Let $\{x_1,\dots,x_n\}$ be the set of generators of the cohomology ring, A be the set of ring relations, $\mathbb{Z}/p\mathbb{Z}$ be the coefficients. Then, we have the isomorphism:*

$$H^*(G,\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}[x_1,\dots,x_n]/\langle A\rangle$$

Theorem 2.2 ([2] Lemma 26.4)**.** *Let G be an elementary abelian 2-group of rank n. The cohomology $H^*(G,\mathbb{Z}/2\mathbb{Z})$ is generated as a ring by a basis $\{x_1,\dots,x_n\}$ for $H^1(G,\mathbb{Z}/2\mathbb{Z})$. The negligible ideal $I(G,\mathbb{Z}/2\mathbb{Z})$ over $\mathbb{Q}$ is generated by elements of the form $\{x_ix_j^2 + x_i^2x_j : 1 \le i \le j \le n\}$.*

In [8], Quillen and Venkov proved that nilpotent elements in mod p cohomology rings of p-groups can be detected on the cohomology of elementary abelian p-subgroups. By an analogous argument in Galois cohomology, Gherman proved the following unpublished generalization:

Theorem 2.3. *Let G be a group. Then $u \in H^*(G,\mathbb{Z}/2\mathbb{Z})$ is eventually negligible over $\mathbb{Q}$ if and only if the restriction of u to each elementary abelian 2-subgroup is eventually negligible.*

Theorems 2.2 and 2.3 will serve as our primary detection method for eventually negligible elements. Our methodology, then, is as follows:

(1) We begin by taking the restriction maps: $H^*(G,\mathbb{Z}/2\mathbb{Z}) \to H^*(E,\mathbb{Z}/2\mathbb{Z})$ for all *maximal* elementary abelian subgroups E of G.
(2) Using Theorems 2.2 and 2.3, we identify eventually negligible classes. Denote by N the ideal generated by all the eventually negligible elements found in this step. We have that $N \subset N(G,\mathbb{Z}/2\mathbb{Z})$ and wish to prove that $N \supset N(G,\mathbb{Z}/2\mathbb{Z})$ to establish equality.
(3) To do this, we quotient by N and consider $\mathbb{Z}/2\mathbb{Z}[x_1,\dots,x_n]/N$.
(4) Looking at elements in the quotient ring, we take the restriction map and check that there are no non-trivial elements remaining. These elements take the form of a polynomial, so we can plug in arbitrary values from $\mathbb{Z}/2\mathbb{Z}$ to the generators $\{x_1,\dots,x_n\}$. By [2] Lemma 26.4, an element is negligible in $H^*(E,\mathbb{Z}/2\mathbb{Z})$ if and only if the corresponding polynomial evaluates to 0 for all $\mathbb{Z}/2\mathbb{Z}$ inputs of $\{x_1,\dots,x_n\}$. Therefore, we want to prove that each non-trivial element of $H^*(G,\mathbb{Z}/2\mathbb{Z})/N$ restricts to a polynomial that does not evaluate to 0 for some choice of inputs for $\{x_1,\dots,x_n\}$ in $H^*(E,\mathbb{Z}/2\mathbb{Z})$. In other words, we prove $N = N(G,\mathbb{Z}/2\mathbb{Z})$ by showing the injectivity of the map induced by restriction on the right-hand side of the diagram below.

$$\begin{array}{ccc} H^*(G,\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^*(G,\mathbb{Z}/2\mathbb{Z})/N(G,\mathbb{Z}/2\mathbb{Z}) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \overline{\text{res}}} \\ H^*(E,\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^*(E,\mathbb{Z}/2\mathbb{Z})/I(E,\mathbb{Z}/2\mathbb{Z}) \end{array}$$

3. Results for 2-groups with mod 2 coefficients

In this section, we outline the computation of the eventually negligible ideal of the graded cohomology ring $H^*(G,\mathbb{Z}/2\mathbb{Z})$ for certain choices of a 2-group G.

3.1. Finite Abelian 2-Groups.

First we will outline some lemmas that will be helpful in simplifying our computations.

Lemma 3.1. *Let G be a finite abelian 2-group, x_j, y_i be generators of the cohomology ring, such that x_j corresponds to a $(\mathbb{Z}/2\mathbb{Z})$ factor and y_i corresponds to a $(\mathbb{Z}/4\mathbb{Z})$ factor. For all $k \geq 3$ and $\ell \geq 1$, such that $k+l=d$, $x_j^k x_i^l = x_j^{d-2}x_i^2$ in $H^*(G,\mathbb{Z}/2\mathbb{Z})/N$.*

Proof. The restriction maps x_j to itself, and $y_i \mapsto x_i^2$

$$x_j y_i(x_j^2 + y_i) \xrightarrow{res} x_j^3 x_i^2 + x_i x_j^4$$

By Theorem 2.2, $x_j y_i(x_j^2 + y_i) \in N(G,\mathbb{Z}/2\mathbb{Z})$

$$\Rightarrow x_j^3 x_i^2 + x_i x_j^4 = 0$$

$$\Rightarrow x_j^3 x_i^2 = x_i x_j^4$$

Consider $x_j^k y_i^l$:

$$x_j^k x_i^l = x_j^{k-1} x_i^{l-4}(x_j x_i^4) = x_j^{k-1} x_i^{l-4}(x_j^3 x_i^2) = x_j^{k+2} x_i^{l-2}$$

Thus, we can perform the step above, pushing the exponent fully into x_j, such that it takes the form $x_j^{d-2}x_i^2$. ☐

Lemma 3.2. *Let G be a finite abelian 2-group, x_i, x_j be generators of the cohomology ring both corresponding to distinct $(\mathbb{Z}/2\mathbb{Z})$ factors. For all $k \geq 2$, $l \geq 1$ such that $k+l=d$, $x_i^k x_j^l = x_i^{d-1}x_j$ in $H^*(G,\mathbb{Z}/2\mathbb{Z})/N$.*

Proof. The restriction maps are the identity (i.e. $x_i \mapsto x_i$). Thus, by Theorem 2.2, $x_i^2x_j + x_ix_j^2$ are in the eventually negligible ideal.

$$\Rightarrow x_i^2x_j + x_ix_j^2 = 0$$
$$\Rightarrow x_i^2x_j = x_ix_j^2$$

Consider $x_i^kx_j^l$:

$$x_i^{k-1}x_j^{l-2}(x_ix_j^2) = x_i^{k-1}x_j^{l-2}(x_i^2x_j) = x_i^{k+1}x_j^{l-1}$$

We can perform the step above to continue pushing the exponent fully into x_i, such that it takes the form $x_i^{d-1}x_j$. □

3.1.1. *Results for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.*

Proposition 3.3. *Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. The eventually negligible ideal is*

$$N(G, \mathbb{Z}/2\mathbb{Z}) = \langle x_2, x_1y_2(x_1^2 + y_2)\rangle.$$

Proof. The ring generators of the cohomology ring are $\{x_1, x_2, y_2\}$. There is one ring relation, $\{x_1^2\}$. There is only one maximal elementary abelian subgroup $E \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of G, for which the restriction map $\text{res} : H^*(G, \mathbb{Z}/2\mathbb{Z}) \to H^*(E, \mathbb{Z}/2\mathbb{Z})$ is given by:

(1) $\text{res}(x_1) = x_1$
(2) $\text{res}(x_2) = 0$
(3) $\text{res}(y_2) = x_2^2$

By definition, we can see that x_2 is eventually negligible. In addition, using Theorems 2.2 and 2.3, we can conclude that $x_1y_2(x_1^2 + y_2) \in H^*(G, \mathbb{Z}/2\mathbb{Z})$ is eventually negligible, since its image under the restriction map is of the form $x_ix_j^2 + x_i^2x_j$, which is contained in $N(E, \mathbb{Z}/2\mathbb{Z})$. Thus we have

$$N = \langle x_2, x_1y_2(x_1^2 + y_2)\rangle \subset N(G, \mathbb{Z}/2\mathbb{Z})$$

We now need to show that there is no choice of c_i, such that the restriction map of the polynomial is not equal to zero or in the ideal. We will consider the two cases: polynomials of odd degree and polynomials of even degree.

Odd degree:

$$\text{res}(c_1x_1^{2k-1} + c_2x_1^{2k-3}y_2) = c_1x_1^{2k-1} + c_2x_1^{2k-3}x_2^2$$

Let $x_1 = 1, x_2 = 0$, then we have $c_1 = 0$.
Let $x_1 = 1, x_2 = 1$, then we have $c_1 + c_2 = 0 \Rightarrow c_1 = c_2 = 0$.

Even degree:

$$\text{res}(c_1x_1^{2k} + c_2y_2^k + c_3x_1^{2k-2}y_1) = c_1x_1^{2k} + c_2x_2^{2k} + c_3x_1^{2k-2}x_2^2$$

Let $x_1 = 0, x_2 = 1$, then we have $c_2 = 0$.
Let $x_1 = 1, x_2 = 0$, then we have $c_1 = 0$.
Let $x_1 = 1, x_2 = 1$, this gives $c_1 + c_2 + c_3 = 0 \Rightarrow c_2 = c_1 + c_3 = 0 + 0 = 0$. Thus, $c_1 = c_2 = c_3 = 0$.
Thus we have shown that $N(G, \mathbb{Z}/2\mathbb{Z}) = \langle x_2, x_1y_2(x_1^2 + y_2)\rangle$. □

Lemma 3.4. *For all $k \geq 3$ and $\ell \geq 1$, $x_1^ky_2^l = x_1^{2k-3}y_2$ in $H^*(G, \mathbb{Z}/2\mathbb{Z})/N$.*

Proof. Since $x_1y_2(x_1^2 + y_2) \in N(G, \mathbb{Z}/2\mathbb{Z})$, we know that this element must map to equal zero. With a slight abuse of notation, we can say $x_1y_2(x_1^2 + y_2) = x_1^3y_2 + x_1y_2^2 = 0$.
Adding the second term of the sum to both sides, we get:

$$x_1^3y_2 + 2(x_1y_2^2) = x_1y_2^2$$

With $\mathbb{Z}/2\mathbb{Z}$ coefficients,

$$x_1^3y_2 = x_1y_2^2.$$ □

Lemma 3.5. *For even degree, polynomials (elements) in the quotient are of the form* $c_1x_1^{2k} + c_2y_2^k + c_3x_1^{2k-2}y_1$ *for* $c_i \in \mathbb{Z}/2\mathbb{Z}$. *For odd degree, polynomials in the quotient are of the form* $c_1x_1^{2k-1} + c_2x_1^{2k-3}y_2$ *for* $c_i \in \mathbb{Z}/2\mathbb{Z}$.

Proof. See Lemma 3.4. □

3.1.2. *Results for* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Proposition 3.6. *Let* $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. *The eventually negligible ideal is*

$$N(G, \mathbb{Z}/2\mathbb{Z}) = \langle x_1, x_2^2x_3 + x_2x_3^2, x_2y_1(x_2^2 + y_1), x_3y_1(x_3^2 + y_1)\rangle$$

Proof. The ring generators are $\{x_1, x_2, x_3, y_1\}$. The ring relations are $\{x_1^2\}$. The elementary abelian subgroup is $E \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the restriction map is given by:

(1) $\mathrm{res}(x_1) = 0$
(2) $\mathrm{res}(x_2) = x_2$
(3) $\mathrm{res}(x_3) = x_3$
(4) $\mathrm{res}(y_1) = x_1^2$

From this, we get:

$$x_3^2x_2 = x_3x_2^2 \quad (1)$$
$$x_3^3y_1 = x_3y_1^2 \quad (2)$$
$$x_2^3y_1 = x_2y_1^2 \quad (3)$$

We will consider the two cases: polynomials of odd degree and polynomials of even degree. Note, we write it in the most simple form by applying Lemma 3.1, 3.2.

Odd degree:

$$\alpha_1x_3^k + \alpha_2x_2^k + \alpha_3x_3^{k-1}x_2 + \alpha_4x_3^{k-2}y_1 + \alpha_5x_2^{k-2}y_1 + \alpha_6x_3^{k-3}x_2y_1$$

Taking the restriction map:

$$\Big\downarrow res$$

$$\alpha_1x_3^k + \alpha_2x_2^k + \alpha_3x_3^{k-1}x_2 + \alpha_4x_3^{k-2}x_1^2 + \alpha_5x_2^{k-2}x_1^2 + \alpha_6x_3^{k-3}x_2x_1^2$$

Let $x_3 = 1$, $x_2 = 0$, $x_1 = 0 \to \alpha_1 = 0$
Let $x_3 = 0$, $x_2 = 1$, $x_1 = 0 \to \alpha_2 = 0$
Let $x_3 = 0$, $x_2 = 1$, $x_1 = 1 \implies \alpha_2 + \alpha_5 = 0 \to \alpha_5 = \alpha_2 = 0$
Let $x_3 = 1$, $x_2 = 0$, $x_1 = 1 \implies \alpha_1 + \alpha_4 = 0 \to \alpha_4 = \alpha_1 = 0$
Let $x_3 = 1$, $x_2 = 1$, $x_1 = 0 \implies \alpha_1 + \alpha_2 + \alpha_3 = 0 \to \alpha_3 = \alpha_1 + \alpha_2 = 0$
Let $x_3 = 1$, $x_2 = 1$, $x_1 = 1 \implies \sum \alpha_i = 0 \to \alpha_6 = 0$ We have proved there contains no nontrivial negligible elements for the odd case. Thus, the odd case is proven.

Even degree:

$$\alpha_1x_3^k + \alpha_2x_2^k + \alpha_3x_3^{k-1}x_2 + \alpha_4x_3^{k-2}y_1 + \alpha_5x_2^{k-2}y_1 + \alpha_6x_3^{k-3}x_2y_1 + \alpha_7y_1^{k/2}$$

Taking the restriction map:

$$\alpha_1x_3^k + \alpha_2x_2^k + \alpha_3x_3^{k-1}x_2 + \alpha_4x_3^{k-2}x_1^2 + \alpha_5x_2^{k-2}x_1^2 + \alpha_6x_3^{k-3}x_2x_1^2 + \alpha_7x_1^k$$

We can repeat the same method as the odd case, except we must add in one extra line.

Let $x_2 = 0$, $x_2 = 0$, $x_1 = 1 \to \alpha_7 = 0$.

This proves the even case as well, thus we have proven there are no nontrivial negligible elements in the quotient, so the negligible ideal contains all candidates. □

3.1.3. *Results for $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.*

Proposition 3.7. *Let $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. The eventually negligible ideal is*

$$N(G, \mathbb{Z}/2\mathbb{Z}) = \langle x_1, x_2, y_1y_2^2 + y_2y_1^2\rangle$$

Proof. The ring generators are $\{x_1, x_2, y_1, y_2\}$. The ring relations are $\{x_1^2, x_2^2\}$. The maximal elementary abelian subgroup is $E \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ The restriction map is:

(1) res(x_1)=0
(2) res(x_2)=0
(3) res(y_1)=x_1^2
(4) res(y_2)=x_2^2

We have $y_1^2y_2 = y_2^2y_1$. We can express any element in the quotient as a polynomial with coefficients α_i. The proof is shown for degree $n = 2k$ as odd degree generators are trivially negligible. We first remark the following lemma:

Lemma 3.8. $y_1^ky_2 = y_1^{k-1}y_2^2 = y_1^{k-2}y_2^3 = ... = y_1y_2^k$

Proof. Trivial using $y_1y_2^2 = y_2y_1^2$ □

$$\alpha_1y_1^k + \alpha_2y_2^k + \alpha_3y_1^{k-1}y_2 + \alpha_4y_1^{k-2}y_2^2 + ... + \alpha_{k+1}y_1y_2^k = 0$$

which, using Lemma 3.8, becomes

$$\alpha_1y_1^k + \alpha_2y_2^k + \alpha_3y_1^{k-1}y_2 + \alpha_4y_1^{k-1}y_2 + ... + \alpha_{k+1}y_1^{k-1}y_2 = \alpha_1y_1^k + \alpha_2y_2^k + (\alpha_3 + \alpha_4 + \alpha_5)y_1^{k-1}y_2 = 0$$

Taking the restriction map yields

$$\downarrow_{res}$$

$$\alpha_1x_1^{2k} + \alpha_2x_2^{2k} + (\alpha_3 + ... + \alpha_{k+1})x_1^{2k-2}x_2^2 = \alpha_1x_1^n + \alpha_2x_2^n + (\alpha_3 + ... + \alpha_{k+1})x_1^{n-1}x_2^2 = 0$$

Let $\alpha_3' = \alpha_3 + \alpha_4 + ... + \alpha_{k+1}$.
This gives $\alpha_1x_1^n + \alpha_2^n + \alpha_3'x_2^{n-1}x_2^2 = 0$
Let $x_1 = 0$ and $x_2 = 0 \Rightarrow \alpha_1 = 0$ and $x_2 = 1$, $x_1 = 0 \Rightarrow \alpha_2 = 0$.
Let $x_1 = x_2 = 1 \Rightarrow \alpha_1 + \alpha_2 + \alpha_3' = 0 \Rightarrow \alpha_3' = 0$.
We have thus proved there are no nontrivial negligible elements in the quotient and

$$N(G, \mathbb{Z}/2\mathbb{Z}) = \langle x_1, x_2, y_1y_2^2 + y_2y_1^2\rangle$$

contains all the negligible elements of $H(G, \mathbb{Z}/2\mathbb{Z})$ □

Remark 3.9. *For any $m \geq 2$, the cohomology of $\mathbb{Z}/2^m\mathbb{Z}$ is the same. A proof of this is given in Sections 3.2 and 3.5 of* [1]. *Although our computations do not involve any finite abelian group with a factor $\mathbb{Z}/2^m\mathbb{Z}$ for $m \geq 3$, this would allow us to generalize our computations to finite abelian groups of higher order.*

Example 3.10. *The cohomology of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ is identical to the cohomology of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.*

3.2. **Dihedral 2-Groups.**

Proposition 3.11. *Let D_{2n} be the dihedral group of order $2n$. The eventually negligible ideal is*

$$N(D_{2n}, \mathbb{Z}/2\mathbb{Z}) = \langle b_{10}b_{11}, b_{10}c_{22}, b_{11}c_{22}\rangle$$

Proof. The ring generators of any 2-Dihedral group are $\{b_{10}, b_{11}, c_{22}\}$. The ring relations are $\{b_{10}b_{11}\}$. There are three elementary abelian subgroups, two of which are maximal [6]. The restriction maps to the two maximal abelian subgroups are as follows:
Restriction map 1:

$$b_{10} \to c_{11}$$
$$b_{11} \to 0$$
$$c_{22} \to c_{10}c_{11} + c_{10}^2$$

Restriction map 2:

$$b_{10} \to 0$$
$$b_{11} \to c_{11}$$
$$c_{22} \to c_{10}c_{11} + c_{10}^2$$

Now, we see that $b_{10}c_{22}, b_{11}c_{22} \in N(G, \mathbb{Z}/2\mathbb{Z})$. Thus, we have the isomorphism:

$$H^*(D_n, \mathbb{Z}/2\mathbb{Z})/N(G, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[b_{10}, b_{11}, c_{22}]/\langle b_{10}b_{11}, b_{10}c_{22}, b_{11}c_{22}\rangle$$

Thus, we know that any element in the quotient homology ring comes in the form of

$$\alpha_1 b_{10}^d + \alpha_2 b_{11}^d + \alpha_3 c_{22}^{d/2}, \text{ such that } \alpha_i \in \mathbb{Z}/2\mathbb{Z}, d \in \mathbb{Z}_+$$

Restriction map 1: $\alpha_1 b_{10}^d + \alpha_2 b_{11}^d + \alpha_3 c_{22}^{d/2} \mapsto \alpha_1 c_{11}^d + \alpha_3(c_{10}c_{11} + c_{10}^2)^{d/2}$
Restriction map 2: $\alpha_1 b_{10}^d + \alpha_2 b_{11}^d + \alpha_3 c_{22}^{d/2} \mapsto \alpha_2 c_{11}^d + \alpha_3(c_{10}c_{11} + c_{10}^2)^{d/2}$
We want to prove that $\alpha_i = 0$ for all i.
Let $c_{10} = 0, c_{11} = 1$:
Restriction map 1 image: $\alpha_1 c_{11}^d = 0$
Restriction map 2 image: $\alpha_2 c_{11}^d = 0$ Since we know that $c_{11} \neq 0$, $\alpha_1 = \alpha_2 = 0$.
Let $c_{10} = 1, c_{11} = 0$:
Restriction map 1 and 2 image: $\alpha_3 c_{10}^{d/2} = 0$ By the same logic $\alpha_3 = 0$ as well. Thus, the only way the image is completely zero, is trivial ($\alpha_i = 0$ for all i). Therefore, the negligible ideal we found contains all negligible elements. □

Remark 3.12. *We produced the results for D_4 (dihedral group with 8 elements), but these results can be generalized to all dihedral 2-groups ($D_8, D_{16}, D_{32}, \dots$), as they share the same ring generators, relations, and restriction maps* [6]. *This is proven algebraically rather than computationally in* [5].

3.3. Semidihedral Group.

Definition 3.13. *Semidihedral 2-groups are defined by the following relations: $SD_{2^n} = \{r, s \mid r^{2^{n-1}} = s^2 = e, srs^{-1} = r^{2^{n-2}-1}\}$*

Proposition 3.14. *The eventually negligible ideal for the semidihedral group of order 16 is*

$$N(SD_{16}, \mathbb{Z}/2\mathbb{Z}) = \langle a_{10}, b_{31}, b_{11} \cdot c_{42}\rangle$$

Remark 3.15. *The cohomology and negligible elements of SD_{16} are the same for any semidihedral group SD_{2^n} of order 2^n, $n \geq 4$.*

Proof. The cohomology of SD_{16} has the following properties [6]:
Ring generators: $\{a_{10}, b_{11}, b_{31}, c_{42}\}$
Ring relations: $\{a_{10}b_{11}, a_{10}^3, a_{10}b_{31}, b_{31}^2 + c_{42}b_{11}^2\}$ The Semidihedral group of order 16 has two maximal elementary abelian subgroups.
These elementary abelian subgroups both share the same restriction maps. Their restriction map is:

(1) $\mathrm{res}(a_{10}) = 0$
(2) $\mathrm{res}(b_{11}) = c_{11}$
(3) $\mathrm{res}(b_{31}) = c_{10}c_{11}^2 + c_{10}^2c_{11}$
(4) $\mathrm{res}(c_{42}) = c_{10}^2c_{11}^2 + c_{10}^4$

Any element / polynomial in this quotient ring can be written as

$$\sum_{k=1}^{M} \alpha_k b_{11}^k + \sum_{l=1}^{N} \beta_l c_{42}^l$$

We consider two cases: if degree is not a multiple of 4 or is a multiple of 4.

If degree is not a multiple of 4 (degree $\neq 0$ mod 4), then there is no second term (since the second term is degree 4). Then, the restriction is taken

$$\alpha b_{11}^k \xrightarrow{res} \alpha c_{11}^k$$

which is not negligible.
If the degree is a multiple of 4, then the restriction is taken as

$$\alpha b_{11}^{4l} + \beta c_{42}^l \xrightarrow{res} \alpha c_{11}^{4l} + \beta(c_{10}^2 \cdot c_{11}^2 + c_{10})^l$$

Let $c_{10} = 0, c_{11} = 1$. Then, $\alpha 1^{4l} + \beta(0^2 \cdot 1^2 + 0)^l = \alpha = 0$. This implies α must equal zero.
Let $c_{10} = 1, c_{11} = 0$. Then, $\alpha 0^{4l} + \beta(1^2 \cdot 0^2 + 1)^l = \beta = 0$. This implies β must also equal zero.
Therefore, there are no other eventually negligible elements in the ideal. □

3.4. **Modular Groups.**

Definition 3.16. *Let $M_{2^n} = \{r, s \mid r^{2^{n-1}} = s^2 = e, srs^{-1} = r^{2^{n-2}-1}\}$ be the modular group of order 2^n.*

Proposition 3.17. *The eventually negligible ideal for the modular group of order 16 is*

$$N(M_{16}, \mathbb{Z}/2\mathbb{Z}) = \langle a_{10}, a_{31}, b_{11}c_{42} \rangle$$

Remark 3.18. *The cohomology and negligible elements of M_{16} are the same as that of any modular group M_{2^n} of order $2^n n \geq 4$.*

Proof. The cohomology ring of M_{16} has the following properties:
Ring generators: $\{a_{10}, b_{11}, a_{31}, c_{42}\}$
Ring relations: $\{a_{10}^2, a_{10}b_{11}^2, a_{10}a_{31,a_{31}^2}\}$ This group only has one restriction map to maximal elementary subgroup:

(1) $\mathrm{res}(a_{10}) = 0$
(2) $\mathrm{res}(b_{11}) = c_{11}$
(3) $\mathrm{res}(a_{31}) = 0$
(4) $\mathrm{res}(c_{42}) = c_{10}^2c_{11}^2 + c_{10}^4$

An element in $H^*(G, \mathbb{Z}/2\mathbb{Z})/N$ takes the form of

$$\alpha_1 b_{11}^k + \alpha_2 c_{42}^l$$

Taking the restriction map, we get the image:

$$\alpha_1 c_{11}^k + \alpha_2 (c_{10}^2 c_{11}^2 + c_{10}^4)^l$$

Let $c_{10} = 0$, $c_{11} = 1$, $\alpha_1 1^k + \alpha_2 (0)^l = \alpha_1 = 0$
Let $c_{10} = 1$, $c_{11} = 0$, $\alpha_1 0^m + \alpha_2 (0 + 1^4)^l = \alpha_2 = 0$
Thus, $\alpha_1 = \alpha_2 = 0$. Therefore, there are no other eventually negligible elements in the eventually negligible ideal. □

3.5. **Quaternion Group.**

Proposition 3.19. *Let $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ be the quaternion group. The negligible ideal for the quaternion group of order 8 is*

$$N(Q_8, \mathbb{Z}/2\mathbb{Z}) = \langle a_{10}, a_{11} \rangle$$

Remark 3.20. *The cohomology and eventually negligible elements of Q_8 are the same as that of any quaternion group Q_{2^n} of order 2^n, $n \geq 3$. This was proven in ([7], Section 1).*

Proof. The ring generators are $\{a_{10}, a_{11}, c_{40}\}$. The ring relations are $\{a_{11}^2 + a_{10}a_{11} + a_{10}^2, a_{10}^3\}$. There is one elementary abelian subgroup, with the restriction map given by:

(1) $\text{res}(a_{10}) = 0$
(2) $\text{res}(a_{11}) = 0$
(3) $\text{res}(c_{40}) = c_{10}^4$

By definition, $N = \langle a_{10}, a_{11} \rangle$. Then, in the quotient ring, we are left with polynomials of the form $\alpha_1 c_{10}^4$ which only trivially evaluates to zero, i.e. only when α_1 or $c_{10} = 0$, so it is not eventually negligible. Thus, we have shown that the only eventually negligible elements are a_{10} and a_{11}, and $N(Q_8, \mathbb{Z}/2\mathbb{Z}) = \langle a_{10}, a_{11} \rangle$. □

Remark 3.21. *The quaternion group (and elementary abelian groups) possess the special property wherein the negligible ideal is equal to the nilradical.*

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