On multinomial identities with integer linear recurrence sequences

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Abstract

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1. Introduction

The Fibonacci sequence, given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for $n \in \mathbb{Z}_{\geq 0}$, has sparked interest since it was discovered, due to its abundant and unexpected identities [8]. It is a special case of an integer linear recurrence sequence, which is a sequence (G_n) satisfying $G_n \in \mathbb{Z}\ \forall n \in \mathbb{Z}_{\geq 0}$ and

$$G_{n+k} = c_{k-1}G_{n+k-1} + c_{k-2}G_{n+k-2} + \cdots + c_0 G_n$$

for all $n \in \mathbb{Z}_{\geq 0}$, where $k \in \mathbb{Z}_{>0}$ and $c_0, \ldots, c_k \in \mathbb{Z}$ are fixed.

Some higher-order generalizations of the Fibonacci sequence were also studied, notably the $k-$bonacci sequence: $F_0^{(k)} = 0$, $F_1^{(k)} = F_2^{(k)} = \cdots = F_{k-1}^{(k)}$, and for $n \geq 0$,

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)}.$$

In [11], R.S. Melham proved novel identities for the tribonacci (3-bonacci) and for the tetranacci (4-bonacci) sequences, denoted by $(p_n)_n$ and $(q_n)_n$, respectively. Some of them are

$$q_{n+6}^2+q_{n+5}^2+2q_{n+4}^2+2q_{n+3}^2-2q_{n+2}^2+q_{n+1}^2-q_n^2 = 46q_{2n}+70q_{2n+1}+82q_{2n+2}+88q_{2n+3}$$

$$p_{m+3}p_{n+3} + p_{m+2}p_{n+2} + p_{m+1}p_{n+1} - p_m p_n = 2p_{m+n} + 3p_{m+n+1} + 3p_{m+n-2}.$$

He also exhibited a method to generate new ones, even for more general integer linear recurrence sequences. In this context, we are interested in determining whether or not identities like these can have arbitrarily large degree.

Previously, authors have shown that under certain conditions, linear recurrence identities cannot have large degree. Marques and Togbé [6] showed that sums of consecutive powers (with exponent at least 3) of Fibonacci numbers cannot fall back in the Fibonacci sequence infinitely many times. Chaves, Marques and Togbé proved that, under certain conditions, sums of powers of elements of integer linear recurrence sequences have bounded degree. We try to extend those boundedness results to identities involving two sequences and multiplication of terms with different indices, building upon previous work in collaboration with Carlos Gustavo Moreira and Ana Paula Chaves.

Our main result is the following:

Theorem 1. *Let M be a positive integer, $(G_n)_n$ and $(H_n)_n$ be integer linear recurrence sequences with nonzero terms and simple dominant roots α and β, respectively. Then, there exists an effectively computable constant E such that the following holds: If $k \in \mathbb{Z}_{>0}$, $s \in \mathbb{Z}_{>0}$, $R(x_0, \ldots, x_k)$ is a multinomial with integer coefficients and degree at most $s-1$, $0 \neq c_0, c_1 \ldots, c_k = 1$ are integers with modulus at most M, and*

$$R(G_{n_i}, \ldots, G_{n_i+k}) + c_0 G_{n_i}^s + \cdots + c_{k-1} G_{n_i+k-1}^s + G_{n_i+k}^s \in (H_n)_n$$

for infinitely many $n_i \in \mathbb{Z}_{>0}$, then s is at most E. (Note here that E does not depend on k)

Here, a linear recurrence sequence $(G_n)_n$ is said to have a dominant root α if α is a root of its characteristic polynomial with modulus bigger than all other roots.

2. Auxiliary results

Lemma 1 (Matveev's theorem). *Let $\alpha_1, \alpha_2, \alpha_3$ be real algebraic numbers and let b_1, b_2, b_3 be non-zero integer rational numbers. Define*

$$\Lambda = \alpha_1^{b_1} \alpha_2^{b_2} \alpha_3^{b_3} - 1.$$

Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$ and let A_1, A_2, A_3 be positive real numbers which satisfy

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\}, \text{ for } j = 1, 2, 3.$$

Assume that $B \geq \max\{|b_j|; 1 \leq j \leq 3\}$. *Define also*

$$C_1 = 1.4 \times 30^6 \times 3^{4.5} \times D^2 \log(eD)$$

If $\Lambda \neq 0$, *then*

$$|\Lambda| > \exp(-C_1 A_1 A_2 A_3 \log(eB)).$$

In the previous statement, the *logarithmic height* of a degree n algebraic number τ is defined as

$$h(\tau) = \frac{1}{n}(\log|a| + \sum_{j=1}^{n} \log\max\{1, |\tau^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of τ (over $\mathbb{Z}$) and $(\tau^{(j)})_{1\leq j\leq n}$ are the (Galois) conjugates of τ.

Lemma 2. *Let* $\alpha > 1$, $g, h \neq 0$ *be algebraic. Let also* $a \in \mathbb{Z}_{>0}$, $\zeta = \alpha^{\frac{1}{a}}$. *Let also* $M > 0$ *be a positive integer. Then there exists an effectively computable constant* E *such that if* $k \in \mathbb{Z}_{>0}$ *and* $c_0, \ldots, c_{k-1}$ *integers with modulus at most* M *satisfying*

$$h\zeta^t = g^s(c_0 \cdots + c_{k-1}\alpha^{s(k-1)} + \alpha^{sk})$$

for some positive integer s *and integer* t, *then* $s < E$.

This is a corollary from the proof of Chaves-Moreira-N. theorem in [12].

3. The proof of Theorem 1

The proof outline is as follows: first, we will give asymptotic estimates for $R(G_{n_i}, \ldots, G_{n_i+k})$, and use Matveev's theorem to prove that β is a positive rational power of α. Then, we will (after some manipulations) take limits as n_i goes to infinity, arriving at a "Diophantine" equation with algebraic numbers. Eliminating one edge case will reduce our main result to the equation in Lemma 2, concluding the proof.

Before delving into specific cases, let's make some remarks on the asymptotic behavior of $(H_n)_n$ and $(G_n)_n$. Let $Q(x_0 \ldots, x_k) = R(x_0, \ldots, x_k) + c_0 x_0^s + \cdots + c_{k-1}x_{k-1}^s + x_k^s$. Write

$$R(G_{n_i}, \ldots, G_{n_i+k}) + c_0 G_{n_i}^s + \cdots + c_{k-1}G_{n_i+k-1}^s + G_{n_i+k}^s = H_{t_i} \tag{1}$$

For some sequences $(n_i)_i$, $(t_i)_i$ of positive integers, and $(n_i)_i$ increasing. We also denote $s = \deg R$. Note that from the theory of linear recurrences, we can write, for some polynomials g_j,

$$G_n = g\alpha^n + \sum g_j(n)\alpha_j^n = g\alpha^n(1 + O(\gamma_1^n)) = g\alpha^n \exp(O(\gamma_1^n))$$

for some positive real $\gamma_1 < 1$, and we assume without loss of generality $\gamma_1 > \frac{1}{\alpha}$. Similarly, if $1 > \gamma > \max(\frac{1}{\alpha}, \gamma_1)$ is chosen to be close enough to 1,

$$H_n = h\beta^n \exp\left(O(\gamma^n)\right) \tag{2}$$

and

$$G_n = g\alpha^n \exp\left(O(\gamma^n)\right). \tag{3}$$

3.1. Proof that β rational power of α

We compute

$$\begin{aligned}
Q(G_{n_i}, \ldots, G_{n_i+k}) &= \sum_{a_0+a_1+\cdots+a_k \leq s} r_{a_0,\ldots,a_k} G_{n_i}^{a_0} \ldots G_{n_i+k}^{a_k} \\
&= \sum_{a_0+a_1+\cdots+a_k \leq s} r_{a_0,\ldots,a_k} \prod_{j=0}^{k} [g^{a_j}\alpha^{a_j(n_i+j)} \exp(O(\gamma^{n_i}))] \\
&= \sum_{a_0+a_1+\cdots+a_k \leq s} r_{a_0,\ldots,a_k} g^{\sum a_j} \alpha^{\sum a_j(n_i+j)} \exp\left(kO(\gamma^{n_i})\right)
\end{aligned}$$

Let r be maximum such that $u = \sum_{a_0+a_1+\cdots+a_k=r} r_{a_0,\ldots,a_k} g^{\sum a_j} \alpha^{\sum a_j j}$ is non-zero. This necessarily exists as if $Q(G_{n_i}, \ldots, G_{n_i+k}) = 0, H_{t_i} = 0$, contradicting our hypothesis.

$$Q(G_{n_i}, \ldots, G_{n_i+k}) = \exp\left(O(\gamma^{n_i})\right)\alpha^{n_i r} \left[\sum_{a_0+a_1+\cdots+a_k=r} (r_{a_0,\ldots,a_k} g^{\sum a_j} \alpha^{\sum a_j j}) + O(\alpha^{-n_i})\right].$$

$$\begin{aligned}
Q(G_{n_i}, \ldots, G_{n_i+k}) &= u\alpha^{n_i r} \exp kO(\gamma^{n_i})(1 + O(\alpha^{-n_i})) \\
&= u\alpha^{n_i r} \exp O(\gamma^{n_i}) \exp(O(\alpha^{-n_i})) \\
&= u\alpha^{n_i r} \exp\left(O(\gamma^{n_i})\right),
\end{aligned}$$

because $\gamma > \frac{1}{\alpha}$. Note that since this is eventually increasing, $H_{t_i} \sim h\beta^{t_i}$ must be eventually increasing, so we can assume without loss of generality that $(t_i)_i$ is increasing.

$$\implies hu^{-1}\beta^{t_i}\alpha^{-n_i r} = \exp(O(\gamma^{n_i})) = 1 + O(\gamma^{n_i}) \tag{4}$$

This gives us

$$t_i = n_i r \frac{\ln \alpha}{\ln \beta} + \frac{\ln(uh^{-1})}{\ln \beta} + O(\gamma^{n_i}) = O(n_i).$$

Then, we apply Matveev's theorem, for $\alpha_1 = hu^{-1}$, $\alpha_2 = \beta$, $\alpha_3 = -n_i r$, $b_1 = 1$, $b_2 = t_i$, $b_3 = -n_i r$, and

$$\Lambda = hu^{-1}\beta^{t_i}\alpha^{-n_i r} - 1.$$

The b_i's are nonzero and the α_i's are algebraic, so this gives us that either $\Lambda = 0$ or there exist constants C_1, A_1, A_2, A_3 such that

$$\begin{gathered}
(eB)^{-C_1 A_2 A_2 A_3} < |\Lambda| = O(\gamma^{n_i}) \\
\implies O(n_i) = eB > O(\gamma^{n_i C_1 A_2 A_2 A_3})^{-1} \\
O(n_i \gamma^{n_i C_1 A_2 A_2 A_3}) > 1,
\end{gathered}$$

which does not hold for large n_i. Hence, we must actually have $h\beta^{t_i} = u\alpha^{n_i r}$ for all sufficiently large i. Dividing the equations for $i+1$ and i, we get $\beta = \alpha^{r\frac{n_{i+1}-n_i}{t_{i+1}-t_i}}$, as desired. In particular, taking $\zeta = \alpha^{\frac{1}{t_{i+1}-t_i}}$ yields that there exist $a, b \in \mathbb{Z}_{>0}$ such that $\alpha = \zeta^a$, $\beta = \zeta^b$.

3.2. Estimating equation (1) *as* $n_i \to \infty$.

In a similar manner as before, we get by denoting $v = (c_0 + c_1\alpha^s + \cdots + c_{k-1}\alpha^{s(k-1)} + \alpha^{sk})g^s$ that

$$Q(G_{n_i}, \ldots, G_{n_i+k}) = \exp\left(O(\gamma^{n_i})\right)\alpha^{n_i s}\left[v + \sum_{a_0+a_1+\cdots+a_k<r} r_{a_0,\ldots,a_k} g^{\sum a_j}\alpha^{\sum a_j(n_i+j)-n_i s}\right].$$

Note that each term of $\sum_{a_0+a_1+\cdots+a_k<s} r_{a_0,\ldots,a_k} g^{\sum a_j}\alpha^{\sum a_j(n_i+j)-n_i s}$ is exponentially decaying, so this goes to zero as $n_i \to \infty$. Equating this estimate for $Q(G_{n_i}, \ldots, G_{n_i+k})$ with our estimate for H_n, we get

$$\exp\left(kO(\gamma^{n_i})\right)\alpha^{n_i s}\left[v + \sum_{a_0+a_1+\cdots+a_k<s} r_{a_0,\ldots,a_k} g^{\sum a_j}\alpha^{\sum a_j(n_i+j)-n_i s}\right] = h\beta^{t_i}\exp\left(O(\gamma^{t_i})\right)$$

Substituting α and β by powers of ζ,

$$\exp\left(kO(\gamma^{n_i})\right)\zeta^{an_is}\left[v+\sum_{a_0+a_1+\cdots+a_k<s} r_{a_0,\dots,a_k} g^{\sum a_j}\alpha^{\sum a_j(n_i+j)-n_is}\right]=h\zeta^{bt_i}\exp\left(O(\gamma^{t_i})\right)$$

$$\zeta^{bt_i-an_is}=\exp\left(kO(\gamma^{n_i})+O(\gamma^{t_i})\right)h^{-1}\left[v+\sum_{a_0+a_1+\cdots+a_k<s} r_{a_0,\dots,a_k} g^{\sum a_j}\alpha^{\sum a_j(n_i+j)-n_is}\right]\to vh^{-1}$$

as $i\to\infty$. If $v=0$,

$$c_0+\cdots+c_{k-1}\alpha^{s(k-1)}=-\alpha^{sk}$$

$$\implies |\alpha^{sk}|\le M(1+\alpha^s+\cdots+\alpha^{s(k-1)})=M\frac{\alpha^{s(k-1)+1}-1}{\alpha-1}<M\frac{\alpha^{s(k-1)+1}}{\alpha-1}$$

$$\alpha^{s-1}<\frac{M}{\alpha-1}$$

$$s\le 1+\log_\alpha\frac{M}{\alpha-1}.$$

Now, if $vh^{-1}\neq 0$, there exists an integer t such that

$$\zeta^t=vh^{-1}.$$

$$h\zeta^t=g^s(c_0+\cdots+c_{k-1}\alpha^{s(k-1)}+\alpha^{sk})$$

Then, we are done by Lemma 2. □

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