Definable Cardinality

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In this essay, which is based on a talk that I gave at the 4th European Set Theory Colloquium in 2023, I will discuss the concept of *definable cardinality* for subsets and quotient spaces of Polish spaces and contrast it with the classical notion of cardinality.

Although this concept can be studied in much more generality, to keep things simple I will assume below that *definable* means *Borel (definable)*.

1 Sets

I will start first by considering Borel sets in Polish spaces. Given two such sets P, Q, we have that P, Q have the same (classical) cardinality, in symbols

$$|P| = |Q|$$

if there is an (arbitrary) bijection

$$f \colon P \rightarrowtail\!\!\!\!\rightarrow Q.$$

Since we are in the category of definable sets, it is natural to also consider definable bijections. We therefore say that P, Q have the same *Borel cardinality*, in symbols

$$|P|_B = |Q|_B,$$

if there is a *Borel* bijection

$$f \colon P \rightarrowtail\!\!\!\!\rightarrow Q.$$

There is a classical result, see, e.g., [K1, 15.6], that shows that these notions coincide:

Theorem 1.1. *For any two Borel sets P, Q in Polish spaces,*

$$|P| = |Q| \iff |P|_B = |Q|_B.$$

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2 Quotient Spaces

The situation is however dramatically different when we study, instead of Borel sets in Polish spaces, quotient spaces X/E, where X is a Polish space and E is a Borel equivalence relation on X.

Since the case when X is countable is trivial, I will assume form now on that *Polish spaces X are uncountable*, thus $|X| = |\mathbb{R}|$. Moreover to keep things focused, I will consider below the case when E is a *countable Borel equivalence relation (CBER)*, i.e., each E-class is countable. A survey of the theory of countable Borel equivalence relations can be found in [K2].

There are many important countable Borel equivalence relations that appear in several areas of mathematics. Here is a small sample:

- Commensurability of positive reals.
- Turing and arithmetical equivalence in $2^{\mathbb{N}}$.
- Equality mod finite in $\mathcal{P}(\mathbb{N})$.
- Isomorphism of torsion-free abelian groups of finite rank, i.e, subgroups of $(\mathbb{Q}^n, +)$, for some $n \geq 1$.
- Orbit equivalence relations induced by Borel actions of countable groups on Polish spaces. (The *orbit equivalence relation* of a group action has as its equivalence classes the orbits of the action. Its quotient space is the *orbit space* of the action.)

In fact we have the following result:

Theorem 2.1 (Feldman-Moore [FM]). *Every countable Borel equivalence relation is generated by a Borel action of a countable group.*

Given two quotients $X/E, Y/F$ of Polish spaces by countable Borel equivalence relations, we say as usual that they have the same (classical) cardinality, in symbols

$$|X/E| = |Y/F|,$$

if there is an (arbitrary) bijection

$$f\colon X/E \rightarrowtail\!\!\!\!\!\twoheadrightarrow Y/F.$$

We next say that they have the same *Borel cardinality*, in symbols

$$|X/E|_B = |Y/F|_B,$$

if there is a bijection
$$f\colon X/E \rightarrowtail\!\!\!\!\twoheadrightarrow Y/F,$$
that has a Borel lifting
$$f^*\colon X \to Y$$
i.e., satisfies
$$f([x]_E) = [f^*(x)]_F.$$
Similarly we define the preorder
$$|X/E|_B \leq |Y/F|_B$$
if there is an injection
$$f\colon X/E \rightarrowtail Y/F,$$
with Borel lifting.

Now it is clear that all quotient spaces X/E, with E a countable Borel equivalence relation, have the same (classical) cardinality
$$|X/E| = |\mathbb{R}|,$$
so the picture of the (classical) cardinalities of these quotient spaces is an uninspiring dot:
$$\bullet|X/E| = |\mathbb{R}|.$$
This fact crucially uses the Axiom of Choice (AC) and it is a very crude way to measure the size of such quotient spaces as it does not take into account the structure of E itself.

For example, is it reasonable to think that there are as many Turing degrees as there are sets of integers mod finite or that all orbit spaces of actions of the free group $\mathbb{F}_2$ have the same cardinality as the orbit spaces of actions of the group $\mathbb{Z}$?

It turns out that studying the *Borel cardinality* of quotient spaces of countable Borel equivalence relations reveals a much more complex and deep structure.

3 Borel Cardinalities of Quotients of CBERs

To discuss this structure, we first need to discard a trivial (in our context) case, that of the *smooth* relations. A countable Borel equivalence relation E

is called *smooth* if it admits a Borel *transversal*, i.e., a Borel set that contains exactly one element from each E-class. For those it is easy to check that

$$|X/E|_B = |\mathbb{R}|_B.$$

(where we identify $\mathbb{R}$ with its quotient by the equality relation). So from now on I will assume that we are *only considering nonsmooth countable Borel equivalence relations*.

Here is then a rough picture of the preorder of Borel cardinalities $|X/E|_B$ of quotients by countable Borel equivalence relations:

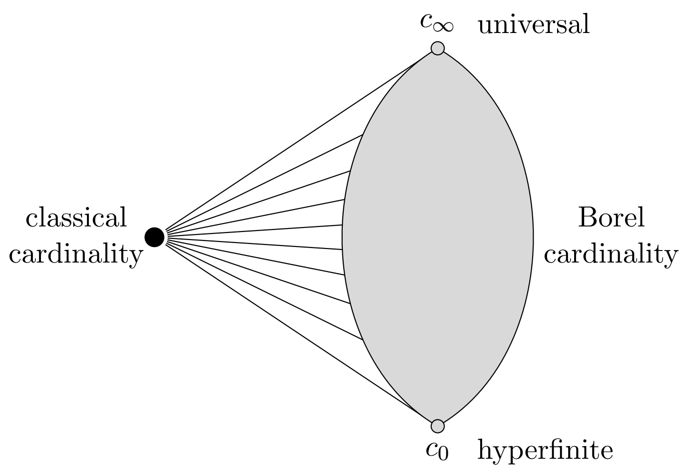

Figure 1: Classical vs Borel cardinalities

Here is an explanation of this picture:

- There is a smallest Borel cardinality, c_0, where

  $$c_0 = |X/E|_B,$$

  for any hyperfinite Borel equivalence E, where E is *hyperfinite* if $E = \bigcup_n E_n$, with $E_1 \subseteq E_2 \subseteq \dots$ Borel equivalence relations with finite equivalence classes. Equivalently these are the CBERs generated by Borel actions of the group $\mathbb{Z}$. In particular this indicates that all the hyperfinite quotients (i.e., quotients by hyperfinite relations) have the same Borel cardinality. Typical examples of hyperfinite quotients are: the orbit space of the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$, the space of subsets of $\mathbb{N}$ mod finite and the isomorphism classes of torsion-free abelian groups of rank 1. See [K2, Chapter 8] for references for these results.

- There is a largest Borel cardinality, c_∞. Typical examples of quotient spaces with this cardinality are: the orbit space of the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ and (Slaman-Steel) the space of all arithmetical degrees; see [K2, Chapter 12].

- $c_0 < c_\infty$ (see [K2, Chapter 6]).

- There are continuum many intermediate Borel cardinalities

$$c_0 < c < c_\infty,$$

  including many incomparable ones (Adams-Kechris [AK]). Typical examples of intermediate cardinalities are those of the orbit spaces of *free* (i.e., having trivial stabilizers) Borel actions of certain countable groups, including free actions of nonamenable groups that admit an invariant Borel probability measure, as for example the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ restricted to its free part. Another class of examples are the isomorphism types of the torsion-free abelian groups of rank n, for $n \geq 2$. For references, see [K2, Chapter 7].

4 Methodology

Next I would like to discuss some important methodological points concerning the study of Borel cardinalities of quotients of countable Borel equivalence relations.

(a) The results mentioned earlier about the structure of Borel cardinalities of quotients X/E, where E is a countable Borel equivalence relation, have been proved using methods of ergodic theory and crucially employ measure theory. By contrast, *generically* all countable Borel equivalence relations are hyperfinite, so on comeager sets their quotient spaces have the same Borel cardinality, c_0, see [K2, Section 8.3].

(b) Therefore, it has been a major open problem to find purely set theoretic methods to prove such results.

(c) Using methods of ergodic theory, it turns out that there are important *set theoretic rigidity phenomena* underlying many of these results. For example, *under certain circumstances*, if E is the orbit equivalence relation induced by a Borel action of a countable group on a Polish space X, then $|X/E|_B$ "remembers" a lot about the group (and sometimes the action).

For instance, if for a set of odd primes S, we let

$$H_S = \bigstar_{p\in S}(\mathbb{Z}/p\mathbb{Z} \star \mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}$$

and let E_S is the equivalence relation induced by the shift action of H_S on 2^{H_S}, restricted to its free part X_S, then

$$|X_S/E_S|_B \leq |X_T/E_T|_B \iff S \subseteq T,$$

(Hjorth-Kechris); see [K2, Theorem 7.11].

Consider next the canonical action of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathbb{R}^n/\mathbb{Z}^n$ and let G_n be the associated orbit equivalence relation. Then

$$m \leq n \iff G_m \leq_B G_n.$$

(see [AK]). In particular, the Borel cardinality of the orbit space of this action "encodes" the dimension n.

For another example, if E_n is the isomorphism relation of torsion-free abelian groups of rank n (on an appropriate Polish space X_n), then $|X_n/E_n|_B$ remembers n:

$$|X_m/E_m|_B \leq |X_n/E_n|_B \iff m \leq n,$$

(Thomas, see [T]). In particular, for $n \geq 2$,

$$c_0 < |X_n/E_n|_B < c_\infty.$$

(d) As seen in (c), it should be emphasized that the existence of intermediate or incomparable Borel cardinalities is not due to the construction of artificial counterexamples (compare this, for example, with existence of incomparable r.e. degrees) but reflects structural differences of natural and important examples.

5 Problems

I will finish by mentioning a few major open problems that have been open for decades.

1. What is the Borel cardinality, c_{TD}, of the set of Turing degrees? It is known that

$$c_0 < c_{TD} \leq c_\infty$$

(see [SlSt]) but whether $c_{TD} = c_\infty$ is unknown. If this was the case, this would contradict Martin's Conjecture on functions on the Turing degrees.

2. Let E be the orbit equivalence relation induced by a Borel action of a countable *amenable* group. Is $|X/E|_B = c_0$, i.e., is E hyperfinite? (Weiss, see [W])

3. Let $E_1 \subseteq E_2 \subseteq \dots$ be hyperfinite countable Borel equivalence relations. Thus for each n, $|X/E_n|_B = c_0$. Let $E = \bigcup_n E_n$. Is E hyperfinite, i.e., $|X/E|_B = c_0$.

4. If $F \subseteq E$ are countable Borel equivalence relations and $|X/F|_B = c_\infty$, is it true that $|X/E|_B = c_\infty$ (Hjorth, see [K2, Problem 12.22]).

References

[AK] A.S. Kechris and S.Adams, Linear algebraic groups and countable Borel equivalence relations, *J. Amer. Math. Soc.*, **13(4)** (2000), 909–943.

[FM] J. Feldman and C.C. Moore, Ergodic equivalence relations and von Neumann algebras, I, *Trans. Amer. Math. Soc.*, **234** (1977), 289–324.

[K1] A.S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, 1995.

[K2] A.S. Kechris, *The Theory of Countable Borel Equivalence Relations*, Cambridge University Press, 2025.

[SlSt] T. A. Slaman and J. R. Steel, Definable functions on degrees, in *Cabal Seminar 81–85*, Lecture Notes in Math., **1333** (1988), Springer-Verlag, 37–55.

[T] S. Thomas, The classification problem for torsion-free abelian groups of finite rank, *J. Amer. Math. Soc.*, **16(1)** (2003), 233–258.

[W] B. Weiss, Measurable dynamics, *Contemp. Math.*, **26** (1984), 395–421.

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